ON THE SHAPE OF NON-MONETARY MEASURES FOR RISKS

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ABSTRACT

This paper investigates how welfare losses for facing risks change as a function of the number of risk exposures. To that aim, we define the risk apportionment of order n (RA-n) utility premium as a measure of pain associated with facing the passage from one risk to a riskier one. Changes in risks are expressed through the specific concept of stochastic dominance of order n defined by Ekern (1980). Three configurations of risk exposures are considered. The paper first shows how the RA-n utility premium is modified when individual's wealth becomes riskier. This makes it possible to generalise earlier results on the topic. Second, the paper provides necessary and sufficient conditions on individual preferences for superadditivity and subadditivity of the RA-n utility premium. Third, the paper investigates welfare changes of merging increases in risks.

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1. INTRODUCTION

The issue of how the presence of multiple risks modify individual behaviour in the face of another risk has been leading to a prolific literature during the last decades.[1] Most of these studies use monetary measures to analyze behaviour towards risk, the most well-known being the risk premium and the willingness to pay. More recently, a few papers have made used of non-monetary measures to provide new behavioural results in the face of risks. In particular, the concept of the utility premium originally introduced by Friedman and Savage (1948) has regained present interest. For instance, Eeckhoudt and Schlesinger (2006) rely on the utility premium to propose a unified approach to explain the meaning of the signs of the successive derivatives of the utility function. Eeckhoudt and Schlesinger (2009) also reexamine the properties of the utility premium and explain the relevance of this tool for decision making. Recently, Crainich and Eeckhoudt (2008) and Courbage and Rey (2010) used non-monetary measures of prudence and temperance to extract behavioural results. Such non-monetary measures not only offer alternative tools to analyse the individual loss of welfare due to the presence of risks, but also allow for much simpler conditions on individual preferences to predict behaviour towards risks.

An issue of importance when dealing with measures of risks is how these measures react to a riskier environment. In particular, knowing how welfare losses of facing increases in risks change as a function of the number of risk exposures offers crucial knowledge on how individuals react to riskier environment. To address these issues, this paper defines the risk apportionment of order n (RA-n) utility premium as a measure of pain associated with facing the passage from one risk to a riskier one. Changes in risks are expressed through the specific concept of stochastic dominance of order n defined by Ekern (1980).

The paper first shows how the RA-n utility premium is modified when individual wealth becomes riskier. It makes it possible to generalise earlier results on the topic, and in particular those of Courbage and Rey (2010). Second, the paper provides necessary and sufficient conditions on individual preferences for superadditivity and subadditivity of the RA-n utility premium. A measure is said to be superadditive or convex in risks if the measured value of two risks is superior to the sum of the values of each risk; the opposite holding for subadditivity. Superadditivity/subadditivity is of relevance when one wants to know how welfare loss changes as a function of the number of risk exposure. This concept sheds light on whether risks are self-aggravating for individuals. For instance, Eeckhoudt and Gollier (2001) look at the superadditivity of three monetary measures of risk, the compensating premium, the risk premium and the willingness to pay. They provide con-

[1] See Eeckhoudt and Gollier (2013) for a review.

ditions on individual preferences for which these measures are supperadditive. They show, in particular, that risk vulnerability (see Gollier and Pratt, 1996) is a sufficient condition for superadditivity of the risk premium. The concept of subadditivity of risks measures has also been popularised through the definition of coherent risk measures defined by Artzner et al. (1999). A measure is said to be coherent if it verifies four axioms among which subadditivity. Third, the paper uses the RA-n utility premium to investigates a related but different issue, that is whether welfare loss changes when increases in risks are merged instead of facing them seperately.

The paper shows that the degree of pain due to facing an increased risk grows when the initial wealth becomes riskier if the signs of the successive derivatives of the utility function alternate in signs, i.e. when preferences exhibit mixed risk-aversion as defined by Caballé and Pomansky (1996). It also shows that risk aversion and temperance are sufficient conditions for superadditivity of the RA-n utility premium. Finally, mixed risk-aversion is shown to drive welfare changes of merging increases in risks.

The paper is organised as follows. Section 2 introduces the benchmark model for non-monetary measures of risk and in particular the RA-n utility premium. Section 3 investigates the effect of riskier initial wealth on the RA-n utility premium. Section 4 addresses the conditions on individuals preferences for superadditivity/subadditivity of non-monetary measures of risks. Section 5 deals with welfare changes of merging increases in risks. Section 6 finally offers a short conclusion.

2. THE BENCHMARK MODEL

2.1. Non-monetary measures in the face of risks

Non-monetary measures in the face of risks stem from the work of Friedman and Savage (1948) who used expected utility theory to define risk aversion and introduced two ways for its measure. The two measures reflect the subjective cost of risk for a risk averter.

Let an individual's final wealth be represented by $x + \tilde{\epsilon}$ where x ($x > 0$) denotes the initial wealth of the individual and $\tilde{\epsilon}$ is a zero-mean random variable[2]. The first measure of risk aversion in the face of the risk $\tilde{\epsilon}$ at wealth level x is a monetary measure, *the risk premium* $\pi(x, \tilde{\epsilon})$, and is such that:

$$E[u(x + \widetilde{\epsilon})] = u(x - \pi(x, \tilde{\epsilon})), \tag{1}$$

where u denotes the individual's von Neumann-Morgenstern utility function (with $u'(x) \geq 0 \; \forall x$) and E denotes the expectation operator. $\pi(x, \tilde{\epsilon})$ is the amount of money that the

[2] We assume that the support of $\tilde{\epsilon}$ is defined such that $x + \epsilon$ is in the domain of u.

agent is ready to pay to get rid of the zero-mean risk $\widetilde{\epsilon}$. $\pi(x,\tilde{\epsilon}) \geq 0$ if and only if the individual is risk-averse ($u''(x) \leq 0\ \forall x$). The second one is a non-monetary measure of risk aversion, *the utility premium*, $w_A(x,\tilde{\epsilon})$:

$$w_A(x,\widetilde{\varepsilon}) = u(x) - E[u(x+\widetilde{\varepsilon})]. \tag{2}$$

$w_A(x,\widetilde{\varepsilon})$ measures the degree of "pain" associated with facing the risk $\widetilde{\varepsilon}$, where pain is measured by the loss in expected utility from adding the risk $\widetilde{\varepsilon}$ to wealth x. From Jensen's inequality, $w_A(x) \geq 0$ if and only if $u''(x) \leq 0\ \forall x$[3]

Prudence is known as preference for a zero-mean risk in the wealthier state of nature. The prudence utility premium as introduced by Crainich and Eeckhoudt (2008), denoted $w_P(x,\widetilde{\varepsilon})$, measures the increase in pain of facing the risk $\widetilde{\varepsilon}$ in the presence of a sure loss $k > 0$. This is defined as follows:

$$w_P(x,\widetilde{\varepsilon}) = u(x-k) - E[u(x-k+\widetilde{\varepsilon})] - (u(x) - E[u(x+\widetilde{\varepsilon})]), \tag{3}$$

which is equivalent to:

$$w_P(x,\widetilde{\varepsilon}) = w_A(x-k,\tilde{\epsilon}) - w_A(x,\tilde{\epsilon}). \tag{4}$$

Naturally, $w_P(x,\widetilde{\varepsilon}) \geq 0$ if and only if $u''' \geq 0$[4].

Temperance is known as preference for disaggregation of two independent zero-mean risks. The temperance utility premium as introduced by Courbage and Rey (2010), denoted $w_T(x,\widetilde{\varepsilon})$, measures the increase in pain of facing the risk $\widetilde{\varepsilon}$ in the presence of an independent zero-mean risk $\tilde{\theta}$. It writes as follows:

$$w_T(x,\widetilde{\varepsilon}) = E[u(x+\tilde{\theta})] - E[u(x+\tilde{\theta}+\widetilde{\varepsilon})] - (u(x) - E[u(x+\widetilde{\varepsilon})]), \tag{5}$$

which is equivalent to:

$$w_T(x,\widetilde{\varepsilon}) = w_A(x+\tilde{\theta},\tilde{\epsilon}) - w_A(x,\tilde{\epsilon}), \tag{6}$$

$w_T(x,\widetilde{\varepsilon}) \geq 0$ if and only if $u^{(4)} \leq 0$[5].

Courbage and Rey (2010) suggested an extension of these measures to higher orders defining the utility premium by iteration following Eeckhoudt and Schlesinger (2006).

[3]We assume throughout this article that the utility function u is nth differentiable. As it is usual, we assume that the derivative of order k ($\forall k \geq 1$), denoted $u^{(k)}(x)$, has a constant sign in the domain of u: $u^{(k)}(x) \geq 0$ or $u^{(k)}(x) \leq 0\ \forall x$. To simplify notations we will only write $u^{(k)} \geq 0$ or $u^{(k)} \leq 0$.

[4]To see this note that, using Eeckhoudt and Schlesinger (2006), Eq. (3) is positive by preference for disaggregation of harms by a prudent individual. Now, prudence defined in this way is equivalent to a positive third derivative of the utility function, by Jensen's inequality.

[5]Note that Eq. (5) is positive iff the individual is temperant by the expected utility equivalence of the definition of temperance in Eeckhoudt and Schlesinger (2006). Now, we see from Eq. (6) that $w_T \geq 0$ is equivalent to w_A is convex by Jensen's inequality. From the definition of w_A in Eq. (2) w_A is convex iff u'' is concave, which is equivalent to $u^{(4)} \leq 0$ by Jensen's inequality again.

Denoting $w_{(2)}(x, \tilde{\varepsilon})$ the Friedman and Savage (1948) utility premium of Eq. (2), we can proceed from their remark by defining for all n even and $n \geq 2$:

$$w_{(n+1)}(x, \tilde{\epsilon}) = w_{(n)}(x - k, \tilde{\epsilon}) - w_{(n)}(x, \tilde{\epsilon})$$

with $k > 0$ and

$$w_{(n+2)}(x, \tilde{\epsilon}) = w_{(n)}(x + \tilde{\theta}_n, \tilde{\epsilon}) - w_{(n)}(x, \tilde{\epsilon}),$$

where $\tilde{\theta}_n$ is an independent random variable (i.e. random variables $\tilde{\epsilon}$, $\tilde{\theta}_2$, $\tilde{\theta}_4$, $\tilde{\theta}_6$, etc, are mutually independent) and such that $E(\tilde{\theta}_n) = 0$. As an illustration, when $n = 2$, $w_{(n+1)}(x, \tilde{\epsilon})$ corresponds to the prudence utility premium, $w_P(x, \tilde{\epsilon})$, and $w_{(n+2)}(x, \tilde{\epsilon})$ corresponds to the temperance utility premium, $w_T(x, \tilde{\epsilon})$.

2.2. The RA-*n* utility premium

While Courbage and Rey (2010) suggested to define utility premia of higher orders by iteration of the previous premia of lower orders and in the context of specific lotteries, in this section we present a very general way to define the utility premium at higher orders using the concept of Ekern's increase in risk of order n (Ekern, 1980).

Let's consider two risky situations: a first situation represented by the random variable Y_1 and a second one represented by the random variable X_1. We assume that X_1 and Y_1 are independent, and that X_1 corresponds to an Ekern's increase in risk of order n from Y_1 ($X_1 \preceq_n Y_1$).

Ekern's increase in n-th order risk is defined as follow. Consider X_1 and Y_1 with F and G, respectively, their two cumulative distribution functions of wealth, defined over a probability support contained within the interval $[a, b]$. Define $F_1 = F$ and $G_1 = G$. Now define $F_{k+1}(z) = \int_a^z F_k(t)dt$ and $G_{k+1}(z) = \int_a^z G_k(t)dt$ for $k \geq 1$. The variable X_1 is an increase in nth-order risk over Y_1 ($X_1 \preceq_n Y_1$) if $F_n(z) \leq G_n(z)$ for all z, $F_k(b) \leq G_k(b)$ for $k = 1, 2, .., n-1$, and $E[(X_1)^k] = E[(Y_1)^k]$ for $k = 1, \ldots, n-1$ (i.e the first $n-1$ moments of X_1 and Y_1 are equal). Ekern's (1980) definition includes the case of mean-preserving increase in risk of Rothshild and Stiglitz (1970) as well as of increase in downside risk defined by Menezes *et al.* (1980) as, respectively, a second-degree and a third-degree increase in risk.

We aim to define the non monetary measure of the cost of facing the risk transition e.g. the passage from Y_1 to X_1. We define the function w as follows:[6]:

$$w(x; Y_1, X_1) = E[u(x + Y_1)] - E[u(x + X_1)]. \tag{7}$$

The function $w(x; Y_1, X_1)$ measures the degree of pain associated with facing the passage from the risk Y_1 to the less favorable one, X_1, when the decision-maker initial wealth

[6]We assume throughout this article that the support of any random variable $\tilde{z}$ is defined such that $x + z$ is in the domain of u.

is x. From Ekern (1980), we know that $w(x; Y_1, X_1) \geq 0$ if and only if $(-1)^{(1+n)} u^{(n)} \geq 0$. We formulate the following definition.

Definition

The function w *defined as* $w(x; Y_1, X_1) = E[u(x + Y_1)] - E[u(x + X_1)]$ *is named the risk apportionment of order* n *utility premium, also denoted the RA-*n *utility premium. It measures the degree of pain due to misapportionment of order* n*.*

Note that $(-1)^{(1+n)} u^{(n)} \geq 0$ means that all odd derivatives of u are positive and all even derivatives of u are negative. Following Brockett and Golden (1987) and according to Cabballé and Pomansky (1996), an individual with such a utility function is said to be *mixed risk-averse*. Hence, for all order n, the RA-n utility premium of a mixed risk averse agent is always positive. In other words, such an individual always incurs a pain when facing an increase in risk.

Particular cases of this premium are the various premia defined in the previous section. For instance, when $Y_1 = 0$ and X_1 is a zero-mean background risk, $X_1 = \tilde{\epsilon}$ with $E(\tilde{\epsilon}) = 0$, the function w writes as $w(x; 0, \tilde{\epsilon}) = u(x) - E[u(x + \tilde{\epsilon})] = w_A(x, \tilde{\varepsilon})$. It is the utility premium introduced by Friedman and Savage (1948) to define the non monetary risk aversion measure: $w(x; 0, \tilde{\epsilon}) \geq 0$ if and only if the individual is risk averse ($u''(x) \leq 0$). When Y_1 and X_1 are defined as equiprobable lotteries describing an increase in downside risk (Menezes et al. (1980)): $Y_1 = [-k, \tilde{\epsilon}; \frac{1}{2}, \frac{1}{2}]$ and $X_1 = [0, -k + \tilde{\epsilon}; \frac{1}{2}, \frac{1}{2}]$ with $k > 0$ and $E(\tilde{\epsilon}) = 0$, the function w writes as

$$w(x; Y_1, X_1) = \frac{1}{2} w_P(x, \tilde{\epsilon}), \tag{8}$$

with $w_P(x, \tilde{\epsilon})$ the prudence utility premium defined by Crainich and Eeckhoudt (2008) which is positive if and only if $u^{(3)} \geq 0$. When Y_1 and X_1 are defined as the following lotteries: $Y_1 = [\tilde{\theta}, \tilde{\epsilon}; \frac{1}{2}, \frac{1}{2}]$ and $X_1 = [0, \tilde{\theta} + \tilde{\epsilon}; \frac{1}{2}, \frac{1}{2}]$, with $\tilde{\theta}$ and $\tilde{\epsilon}$ independent and zero mean random variables ($E(\tilde{\theta}) = E(\tilde{\epsilon}) = 0$), the function w writes as

$$w(x; Y_1, X_1) = \frac{1}{2} w_T(x, \tilde{\epsilon}), \tag{9}$$

with $w_T(x, \tilde{\epsilon}) = E[u(x + \tilde{\theta})] + E[u(x + \tilde{\epsilon})] - u(x) - E[u(x + \tilde{\theta} + \tilde{\epsilon})]$, that corresponds to the temperance utility premium defined by Courbage and Rey (2010) which is positive if and only if $u^{(4)}(x) \leq 0$.

3. RA-n UTILITY PREMIUM AND INCREASES IN RISKS

As Courbage and Rey (2010), we can also investigate, in this more general context of the RA-n utility premium, how this measure reacts to the introduction on wealth of a sure loss or a zero-mean background risk. As intuition suggests, the pain increases in both cases under usual conditions on the signs of higher-orders derivatives of the utility function.

Indeed, rewriting the impact of a sure loss and a background risk using the RA-n utility premium, we obtain:

$$w(x; Y_1, X_1) - w(x-k; Y_1, X_1) \leq 0 \Leftrightarrow (-1)^{(n)} u^{(n+1)} \geq 0, \quad (10)$$

$$w(x; Y_1, X_1) - w(x+\tilde{\epsilon}; Y_1, X_1) \leq 0 \Leftrightarrow (-1)^{(n+1)} u^{(n+2)} \geq 0. \quad (11)$$

This result is rather easy to obtain. Eq. (10) rewrites as $E[u(x+Y_1)] - E[u(x+X_1)] \leq E[u(x-k+Y_1)] - E[u(x-k+X_1)]$ which is equivalent to $E[u(x+Y_1)] + E[u(x-k+X_1)] \leq E[u(x+X_1)] + E[u(x-k+Y_1)]$. Using Eeckhoudt et al. (2009), we obtain that this inequality holds iff $(-1)^{(n)} u^{(n+1)} \geq 0$. In the same vein, Eq. (11) rewrites as $E[u(x+Y_1)] - E[u(x+X_1)] \leq E[u(x+\tilde{\epsilon}+Y_1)] - E[u(x+\tilde{\epsilon}+X_1)]$ which is equivalent to $E[u(x+Y_1)] + E[u(x+\tilde{\epsilon}+X_1)] \leq E[u(x+X_1)] + E[u(x+\tilde{\epsilon}+Y_1)]$. Using Eeckhoudt et al. (2009), we obtain that this inequality holds iff $(-1)^{(n+1)} u^{(n+2)} \geq 0$.

Eqs. (10) and (11) respectively mean that $(-1)^{(n)} u^{(n+1)} \geq 0$ is equivalent to the RA-n utility premium being vulnerable to a sure loss and that $(-1)^{(n+1)} u^{(n+2)} \geq 0$ is equivalent to the RA-n utility premium being vulnerable to a zero-mean background risk. Results of Eq . (10) and (11) respectively also mean that the RA-n utility premium is a decreasing and convex function in x, the wealth level. This is rather intuitif as when the individual gets richer the pain of facing increases in risks is reduced as he can deal with them more easily, but this reduction in pain diminishes as the individual's wealth increases. Hence, a positive, respectively negative, sign of the derivative of order k when k is even, respectively odd, means that the RA-$(k-1)$ utility premium is decreasing in x and the RA-$(k-2)$ utility premium is convex in x.

The concept of the RA-n utility premium makes it possible to extend such analysis to a more general context. The question is then to measure the degree of pain associated with facing the passage from Y_1 to X_1 when the wealth level becomes riskier. A riskier wealth corresponds to the random wealth level, initially equalled to $x + Y_2$ becoming $x + X_2$ where X_2 is an Ekern's increase in risk of order s from Y_2 ($X_2 \preceq_s Y_2$ with X_2 and Y_2 being independent random variables). The degree of pain associated with facing the passage from Y_1 to X_1 when the initial wealth level becomes riskier is defined by the following expression[7]:

$$w(x+X_2; Y_1, X_1) - w(x+Y_2; Y_1, X_1). \quad (12)$$

A positive sign of (12) means that the pain facing the passage from Y_1 to X_1 increases when the wealth level becomes riskier. We obtain the following proposition.

[7]Note that Eqs. (4) and (6) correspond to cases where $Y_2 = 0$ and $X_2 = -k$ ($k > 0$) or $X_2 = \widetilde{\theta}$ ($E(\widetilde{\theta}) = 0$).

Proposition 1.

Given mutually independent random variables X_1, Y_1, X_2, and Y_2, such that $X_1 \preceq_n Y_1$ and $X_2 \preceq_s Y_2$ $w(x+X_2;Y_1,X_1) \geq w(x+Y_2;Y_1,X_1)$ for all utility functions u if and only if the utility function u is such that $(-1)^{(1+n+s)}u^{(n+s)} \geq 0$.

Proof. Using the definition of w, $w(x+X_2;Y_1,X_1) - w(x+Y_2;Y_1,X_1) \geq 0$ rewrites equivalently as $E[u(x+Y_1+X_2)] - E[u(x+X_1+X_2)] \geq E[u(x+Y_1+Y_2)] - E[u(x+X_1+Y_2)]$, that is equivalent to $E[u(x+Y_1+X_2)] + E[u(x+X_1+Y_2)] \geq E[u(x+Y_1+Y_2)] + E[u(x+X_1+X_2)]$. Following Eeckhoudt et al. (2009), this last expression is equivalent to $(-1)^{(1+n+s)}u^{(n+s)} \geq 0$ that ends the proof.

From this proposition, we can extrapolate the following corollary

Corollary 1.

The four following items are equivalent:

(1) risk apportionment of order S holds,

(2) the pain due to misaportionment of order $(S-1)$ is vulnerable to a sure loss,

(3) the pain due to misaportionment of order $(S-2)$ is vulnerable to a zero-mean background risk,

(4) the pain due to misaportionment of order $(S-s)$ is vulnerable to an increase in risk of order s.

This result offers an alternative interpretation of the sign of the utility function S-order derivative ($u^{(S)}$) which can be easily understood and remembered, without reference to any specific decision problem.

For instance in the case of the utility premium for which $Y_1 = 0$ and $X_1 = \tilde{\epsilon}$ in Eq. (12), Proposition 1 tells us that the degree of pain of facing the risk $\tilde{\epsilon}$ increases when initial wealth becomes risky, i.e. when the zero-mean risk X_2 is added to initial wealth, for a temperant individual. In the same way, in the case of the prudence premium, for which X_1 represents an increase in downside risk over Y_1, Proposition 1 tells us that the degree of pain of facing an increase in downside risk increases when initial wealth becomes risky, i.e. when the zero-mean risk X_2 is added to initial wealth, for an individual featuring edginess.[8]

[8] The concept of edginess, i.e. $u^{(5)} \geq 0$, was introduced by Lajeri-Chaherli (2004) to explain the effects of background risks on precautionary savings.

4. SUPERADDITIVITY OF THE RA-n UTILITY PREMIUM

An issue of importance when dealing with measures of risks is that of superadditivity or subadditivity of these measures. From the risk theory literature (see for example Buhlmann (1985) or Gerber and Goovaerts (1981)), it is well-known that financial risks are very often self-aggravating. This would suggest that the cost of risk for two independent risks should be greater than the sum of costs of the two risks taken in isolation. If it were the case, the cost of risk should be superadditive. Eeckhoudt and Gollier (2001) examine this issue when the cost of risk is defined in terms of risk premium. They show that the risk premium is superadditive if risk aversion is risk vulnerable[9].

We first address the issue of supperadditivity when the cost of risk is defined in non-monetary terms through the concept of the utility premium. The definition of supperadditivity is the following. A real-valued function f is superadditive if $f(n_1 + n_2)$ is larger than $f(n_1) + f(n_2)$ for all $n_1 > 0$ and $n_2 > 0$. The opposite inequality holding true for subadditivity.

We show that the utility premium is supperadditive if and only if the utility function is temperant. This gives the following proposition.

Proposition 2.

Given mutually independent zero-mean risks X_1 and X_2, the utility premium is supperadditive, i.e. $w(x; 0, X_1 + X_2) \geq w(x; 0, X_1) + w(x; 0, X_2)$ if and only if the utility function is such that $u^{(4)} \leq 0$.

Proof. Supperadditivity of the utility premium rewrites as $u(x) - E[u(x+X_1+X_2)] \geq u(x) - E[u(x + X_1)] + u(x) - E[u(x + X_2)]$ which is equivalent to $w_T \geq 0$, i.e. $u^{(4)} \leq 0$. ■

Proposition 2 states that the pain of facing two risks simultaneously is higher than the sum of the pains of facing each risk separately for a temperant individual. Note that while risk vulnerability is required for monetary measures of risk to be supperadditive, for which temperance is a necessary condition, temperance is sufficient to obtain supperaditivity in the case of the utility premium.

Extending supperadditivity to the RA-n utility premium, it writes as

$$w(x; Y, X_1 + X_2) \geq w(x; Y, X_1) + w(x; Y, X_2) \tag{13}$$

[9] Risk vulnerability means that risk aversion increases with the presence of an independent background risk (Gollier and Pratt, 1996). Sufficient and necessary conditions on the utility function to have risk vulnerability are quite complex. A necessary condition for risk vulnerability is $u^{(4)} \leq 0$.

Subaddivity of the RA-n utility premium corresponds naturally to the opposite inequalities.

We obtain that the RA-n utility premium is superadditive for zero-mean risks if the individual is risk-averse and temperant as exhibited in the following proposition.

Proposition 3

Given mutually independent zero-mean risks, the RA-n utility premium is supperadditive, i.e $w(x;Y,X_1+X_2) \geq w(x;Y,X_1)+w(x;Y,X_2)$ *if* $u''0$ *and* $u^{(4)} \leq 0$.

Proof: $w(x;Y,X_1+X_2) > w(x;Y,X_1)+w(x;Y,X_2)$ is equivalent to $E[u(x+X_1)] + E[u(x+X_2)] > E[u(x+X_1+X_2)] + E[u(x+Y)]$. If u" < 0 then $E[u(x+Y)] < u(x)$ and then $E[u(x+X_1+X_2)]+E[u(x+Y)] < E[u(x+X_1+X_2)]+u(x)$. If $u^{(4)} < 0$ then $E[u(x+X_1+X_2)] + u(x) < E[u(x+X_1)] + E[u(x+X_2)]$. Consequently, if u" < 0 and $u^{(4)} < 0$ then $E[u(x+X_1)] + E[u(x+X_2)] > E[u(x+X_1+X_2)] + E[u(x+Y)]$.

According to Proposition 3, the pain of facing an increase in two risks simultaneously is higher than the sum of the pains of facing an increase in each risk separately for a risk-averse and temperant individual.

Note that Proposition 3 imposes no further conditions on the links between the risks, i.e. on the nature of the change in risk between Y and respectively X_1 and X_2. It should also be stressed that temperance only is not a sufficient condition for supperadditivity of the RA-n utility premium. We also need the individual to be risk averse. Hence contrary to the results of Proposition 2 where the utility premium is supperadditive for a risk-lover and temperant individual, a risk lover, even temperant, will never have a supperadditive RA-n utility premium.

5. MERGING INCREASES IN RISKS

We know from Proposition 2 that whenever $u^{(4)} \leq 0$ then the Friedman and Savage (1948) utility premium is superadditive i.e. $w(x;0,X_1+X_2) \geq w(x;0,X_1)+w(x;0,X_2)$. This means that the non monetary cost of the total risk (X_1+X_2) is greater than the sum of the non monetary cost of each risk X_1 and X_2.

In the previous section, we generalised the concept of superadditivity of the RA-n utility premium by considering an increase in risk through the passage from Y to X_i

$(i = 1, 2)$. In this section, we consider the increase in risk as the passage from Y_i with $E(Y_i) = 0$ to X_i with $X_1 \preceq_n Y_1$ and $X_2 \preceq_s Y_2$.

In other words, we wonder under wich conditions on the utility function u the non monetary cost of the total increase in risk (passage from $(Y_1 + Y_2)$ to $(X_1 + X_2)$) is larger than the sum of the non monetary cost of each increase in risk (passage from Y_1 to X_1 and simultaneously passage from Y_2 to X_2). More formally our question is the following. What are properties of u ensuring the following inequality :

$$w(x; Y_1 + Y_2, X_1 + X_2) \geq w(x; Y_1, X_1) + w(x; Y_2, X_2) \tag{14}$$

The following proposition provides conditions for such a comparison.

Proposition 4

Given mutually independent random variables X_1, X_2, Y_1 and Y_2, such that $X_1 \preceq_n Y_1$, $X_2 \preceq_s Y_2$, with $E(Y_1) = E(Y_2) = 0$ then $w(x; Y_1 + Y_2, X_1 + X_2) \geq w(x; Y_1, X_1) + w(x; Y_2, X_2)$, for all utility function u such that $(-1)^{(1+n+s)}u^{(n+s)} \geq 0$ and $(-1)^{(1+n)}u^{(n+2)} \geq 0$ and $(-1)^{(1+s)}u^{(s+2)} \geq 0$.

Proof.

Since $E(Y_1) = 0$ we have $Y_1 \preceq_2 0$. Given that $X_2 \preceq_s Y_2$ and applying Eeckhoudt et al. (2009) theorem, we know that $E[u(x+Y_2)] + E[u(x+X_2+Y_1)] - E[u(x+X_2)] - E[u(x+Y_1+Y_2)] \leq 0$ for all u such that $(-1)^{(1+s)}u^{(s+2)} \geq 0$.

Also, since $E(Y_2) = 0$ we have $Y_2 \preceq_2 0$. Given that $X_1 \preceq_n Y_1$ and applying Eeckhoudt et al. (2009) theorem, we know that $E[u(x+Y_1)]+E[u(x+X_1+Y_2)]-E[u(x+X_1)]-E[u(x+Y_1+Y_2)] \leq 0$ forall u such that $(-1)^{(1+n)}u^{(n+2)} \geq 0$. However, Eeckhoudt et al. (2009) theorem gives $E[u(x+Y_1+Y_2)]+E[u(x+X_1+X_2)]-E[u(x+X_1+Y_2)]-E[u(x+Y_1+X_2)] \leq 0$ for all u such that $(-1)^{(1+s+n)}u^{(s+n)} \geq 0$.

Thus, if u verifies $(-1)^{(1+s)}u^{(s+2)} > 0$, $(-1)^{(1+n)}u^{(n+2)} \geq 0$ and $(-1)^{(1+s+n)}u^{(s+n)} \geq 0$ then the following inequality holds: $\Big(\mathrm{E[u(x+Y_2)]} + E[u(x + X_2 + Y_1)] - E[u(x + X_2)] - E[u(x+Y_1+Y_2)]\Big) + \Big(E[u(x+Y_1)]+E[u(x+X_1+Y_2)]-E[u(x+X_1)]-E[u(x+Y_1+Y_2)]\Big) + \Big(E[u(x + Y_1 + Y_2)] + E[u(x + X_1 + X_2)] - E[u(x + X_1 + Y_2)] - E[u(x + Y_1 + X_2)]\Big) \leq 0$. It rewrites equivalently as: $E[u(x + Y_1)] - E[u(x + X_1)] + E[u(x + Y_2)] - E[u(x + X_2)] \leq E[u(x + Y_1 + Y_2)] - E[u(x + X_1 + X_2)$ that is equivalent to $w(x; Y_1, X_1) + w(x; Y_2, X_2) \leq w(x; Y_1 + Y_2, X_1 + X_2)$ that ends the proof.

The result of Proposition 4 means that welfare is reduced by merging increases in risks instead of facing them separately, i.e. the welfare loss of both increases in risks taken together is larger than the sum of welfare losses from assuming each increase in risk separately.

Note, however, that the two risk sets are considered in isolation. In Eq. (14), on the RHS, there is no link between (Y_1, X_1) on one hand and (Y_2, X_2) on the other hand. Now, these two risk sets are present in the DM's environment. A change of perspective arises if we take this fact into account, and consider that risk 2 is a background risk for the management of risk 1, and inversely. Interestingly, it turns out that this change of perspective reverses the result. For example, in the case of the Friedman-Savage utility premium, we obtain the following proposition:

Proposition 5.

Given mutually independent zero-mean random variables, X_1 and X_2, $w(x; 0, X_1 + X_2) \leq w(x + X_2; 0, X_1) + w(x + X_1; 0, X_2)$ for all utility function u such that $u^{(4)} \leq 0$.

Proof.

The inequality $w(x; 0, X_1 + X_2) \leq w(x + X_2; 0, X_1) + w(x + X_1; 0, X_2)$ rewrites as $\Big(E[u(x + X_2)] - E[u(x + X_2 + X_1)]\Big) + \Big(E[u(x + X_1)] - E[u(x + X_1 + X_2)]\Big) \geq u(x) - E[u(x + X_1 + X_2)]$. Following Eeckhoudt et al. (2009) theorem, this inequality holds for all utility function u verifying $u^{(4)} \leq 0$ that ends the proof.

Thus, we obtain that merging the risks that were present, but in the background of each other, increases welfare.

The result can also be generalized, yielding the following proposition:

Proposition 6

Given mutually independent random variables X_1, X_2, Y_1, and Y_2, such that $X_1 \preceq_n Y_1$ and $X_2 \preceq_s Y_2$, then for all utility function u such that $(-1)^{(1+n+s)} u^{(n+s)} \geq 0$:

(a) $w(x; Y_1 + Y_2, X_1 + X_2) \leq w(x + X_2; Y_1, X_1) + w(x + X_1; Y_2, X_2)$,

(b) $w(x; Y_1 + Y_2, X_1 + X_2) \geq w(x + Y_2; Y_1, X_1) + w(x + Y_1; Y_2, X_2)$.

Proof.

Item (a) rewrites as: $E[u(x + Y_1 + Y_2)] - E[u(x + X_1 + X_2)] \geq \Big(E[u(x + X_2 + Y_1)] - E[u(x + X_2 + X_1)]\Big) + \Big(E[u(x + X_1 + Y_2)] - E[u(x + X_1 + X_2)]\Big)$ that is equivalent to $E[u(x + Y_1 + Y_2)] + E[u(x + X_1 + X_2)] \leq E[u(x + Y_2 + X_1)] + E[u(x + Y_1 + X_2)]$. Following Eeckhoudt et al. (2009) theorem, this inequality holds for all u such that $(-1)^{(1+n+s)} u^{(n+s)} > 0$.

Item (b) rewrites as: $E[u(x + Y_1 + Y_2)] - E[u(x + X_1 + X_2)] \leq \Big(E[u(x + Y_2 + Y_1)] - E[u(x + Y_2 + X_1)]\Big) + \Big(E[u(x + Y_1 + Y_2)] - E[u(x + Y_1 + X_2)]\Big)$ that is equivalent to

$E[u(x+Y_1+Y_2)]+E[u(x+X_1+X_2)] \leq E[u(x+Y_2+X_1)]+E[u(x+Y_1+X_2)]$ as in the case of item (a).

Note that Proposition 5 is a special case of item (a) of Proposition 6 and Proposition 2 is a special case of item (b). Indeed, if we pose $Y_1 = Y_2 = 0$ (that implies considering $n = s = 2$ and $E(X_1) = E(X_2) = 0$), item (b) rewrites as $w(x; 0, X_1+X_2) \geq w(x; 0, X_1) + w(x; 0, X_2)$ that is the result of Proposition 2, and item (a) rewrites as $w(x; 0, X_1+X_2) \leq w(x+X_2; 0, X_1) + w(x+X_1; 0, X_2)$ that is the result of Proposition 6.

The difference between items (a) and (b) arises from background risk considerations on the RHS. In item (a), the DM is aware that risk Y_2 will be replaced by risk X_2 when she feels the loss of welfare from facing the risk X_1 instead of Y_1. In item (b), the DM is blind to this risk substitution. She feels a reduced loss from the sum of individual risk substitutions because she ignores that risk Y_2 will be replaced by risk X_2 when dealing with risk 1, and she ignores that risk Y_1 will be replaced by risk X_1 when dealing with risk 2. In this sense, item (a) reflects a kind of rational expectations, whereas item (b) reflects blindness.

6. CONCLUSION

The paper provides a generalization of non-monetary measures of risk by introducing the concept of risk apportionment of order n (RA-n) utility premium. This measure reflects the degree of pain due to facing the transition from one risk to a more severe one. Changes in risks are expressed through the concept of nth degree increase in risk defined by Ekern (1980). Risk apportionment is taken as a starting point using the definitions of attitudes towards risk introduced by Eeckhoudt and Schlesinger (2006). The prudence utility premium and the temperance utility premium are special cases of our RA-n utility premium.

We first show that the RA-n utility premium increases when the decision-maker faces a riskier wealth under mixed risk aversion. We then turn to the issue of deciding whether the RA-n utility premium is subadditive or superadditive, i.e., deciding whether the cost of an increase in several risks faced jointly is smaller or larger than the sum of these increases in risks faced independently. We obtain that the utility premium for an individual with no initial risk is superadditive if the fourth derivative of the utility function is negative (the decision-maker is temperant). We further show that the more general RA-n utility premium is superadditive if the decision-maker is both risk averse and temperant. We finally adress a related but different issue which is whether it is valuable to merge risks instead of facing them in separate entities. Our results show that an individual whose preferences are mixed-risk averse will have a RA-n utility premium that exhibits all of the properties just quoted. As all commonly used utility functions in economic theory, with the first derivative being positive and the second one being positive, exhibit mixed risk aversion, our results then apply to most individuals facing higher number of risk exposures.

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