On ambiguity apportionment

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Abstract

This paper investigates the notion of changes in ambiguity over loss probabilities in the smooth ambiguity model developed by Klibanoff, Marinacci and Mukerji (2005). Changes in ambiguity over loss probabilities are expressed through the specific concept of stochastic dominance of order n defined by Ekern (1980). We characterize conditions on the function capturing attitudes towards ambiguity under which an individual always considers one situation to be more ambiguous than another in a model of two states of nature. We propose an intuitive interpretation of the properties of this function in terms of preferences for harms disaggregation over probabilities, also labelled ambiguity apportionment.

JEL: D81, H50

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1 Introduction

Since the seminal experiments of Ellsberg (1961), it is well recognized that individuals are averse to ambiguity over probabilities. Ellsberg showed that individuals usually prefer gambles with known rather than unknown probabilities, that is, they are ambiguity averse. Many other experiments have confirmed Ellsberg's work since then (e.g. Chow and Sarin, 2001), and several decision models have been proposed to integrate ambiguity preferences in the face of risky situations (e.g. Gilboa and Schmeidler, 1989; Epstein and Schneider, 2003; Klibanoff, Marinacci and Mukerji, 2005).

A recent stream of literature addresses the effect of ambiguity aversion on economic decisions, whether these are insurance decisions (Alary et al., 2013; Gollier, 2014), medical decisions (Berger et al., 2013), prevention decisions (Snow, 2011), portfolio decisions (Gollier, 2011), or decisions over the value of statistical life (Treich, 2010). Ambiguity aversion is defined as a preference for non-ambiguous situations over ambiguous situations. However, a more general question arises as to how an individual compares two situations of ambiguity over probabilities. More precisely, when can we say that one situation is considered as more ambiguous than another? The aim of this paper is to offer a response to this question in the specific case of two states of nature.

We investigate the notion of changes in ambiguity using the recent theory of ambiguity axiomatized by Klibanoff, Marinacci and Mukerji (2005) (hereafter KMM). Their approach separates ambiguity preferences from risk preferences. It also introduces a simple way to define ambiguity aversion, which is captured through the idea of aversion to any mean-preserving spread in the space of probabilities. This comes from the fact that the introduction of ambiguity constitutes a mean-preserving spread in the space of probabilities.

Consider a probability of loss. In the absence of ambiguity, the decision-maker knows the value of this probability, but is uncertain about its value when ambiguity is present. Uncertainty about the loss probability is represented by a probability distribution over this loss probability. We define changes in ambiguity over probabilities through the specific concept of stochastic dominance of order n defined by Ekern (1980). This approach makes it possible to define a statistical link between the probability distributions capturing the level of ambiguity over the loss probability. It also makes it possible to link the notion of changes in ambiguity to the properties of the function capturing the individual attitudes towards ambiguity, and in particular, to the signs of the successive derivatives of this function. These properties, referred to as ambiguity apportionment, are interpreted in terms of preferences for harms disaggregation over probabilities in a similar way as the ones developed by Eeckhoudt and Schlesinger (2006) in the expected utility theory to explain the meaning of the signs of the successive derivatives of the utility function in terms of preferences for harms disaggregation over wealth.

Changes in ambiguity have been recently addressed in the literature either in the KMM model or in a more general framework. Snow (2010, 2011) and Huang et al. (2015) express increases in ambiguity in the KMM model either in terms of mean-preserving spread or mean-variance-preserving spread of the loss probability and in relation to only ambiguity aversion or ambiguity prudence in the sense that more ambiguity makes an ambiguity-averse or ambiguity-prudent individual worse off. Changes in ambiguity have also recently been defined by Jewitt and Mukerji (2014) in a more general environment also in relation to ambiguity aversion. Our definition of change in ambiguity stems from these works and considers different orders of stochastic dominance and other properties of the function capturing the individual attitudes towards ambiguity to have more ambiguity making an ambiguity-averse individual worse off. Our work should also be related to Baillon (2015) who introduces the concept of ambiguity apportionment. We provide a different interpretation of ambiguity apportionment than his in terms of preferences for harms disaggregation over probabilities in the specific model of KMM. While Baillon (2015) establishes which specifications of widely used ambiguity models imply ambiguity apportionment, we introduce the concept of ambiguity apportionment to express preference for one ambiguous situation over another in the specific ambiguity model of KMM.

This paper is organized as follows. In the next section, we introduce the model of ambiguity aversion. We define the notion of one situation being considered as more ambiguous than another in Section 3. We then propose the concept of change in ambiguity in terms of ambiguity apportionment in Section 4. Finally, a short conclusion is provided in the last section.

2 The benchmark model

Let us consider an individual with an initial wealth w and confronted with two states of nature, a good state that occurs with probability $(1-p)$ and a bad state that occurs with probability p (such that $0 < p < 1$). The individual expected utility is written as

$$V_0(w,p) = (1-p)u^G(w) + pu^B(w) \tag{1}$$

where $u^{G''}(x) < 0 < u^{G\prime}(x)\ \forall x$, $u^{B''}(x) < 0 < u^{B\prime}(x)\ \forall x$ and $u^G(x) > u^B(x)\ \forall x$. Utility functions u^G and u^B can be either state-independent[1] or state-dependent as in the value of a statistical life literature (Drèze, 1962) or as in models of irreplaceable commodity (Cook and Graham, 1977) widely used in the health economics literature.

So as to introduce ambiguity and following Treich (2010), Snow (2010) and Berger et al. (2013), we add a parameter ϵ on the probability of the bad state of nature. This

[1] i.e. $u^G(w) = u(w)$ and $u^B(w) = u(w-L)$ with $L > 0$ and $u'' < 0 < u'$ as in the classical model with a monetary loss L

probability of the bad state of nature, $p+\epsilon$, is ambiguous in the sense that the parameter ϵ is not known precisely and takes on values in $[\underline{\epsilon}, \overline{\epsilon}]$. The ambiguity takes the form of a probability distribution for ϵ. We denote the associated random variable by $\tilde{\epsilon}$. The probability of the bad state of nature writes then $\tilde{p} = p + \tilde{\epsilon}$. Obviously, we assume that the realizations of the random variable $\tilde{p} = p + \tilde{\epsilon}$ belong to]0,1[, i.e. for all ϵ in $[\underline{\epsilon}, \overline{\epsilon}]$, $p+\epsilon$ verifies $0 < p + \epsilon < 1$.

In our benchmark model, risk arises because the decision-maker does not know in which state of nature he will be, the bad state with utility level $u^B(w)$ or the good state with utility level $u^G(w)$. Ambiguity arises because the decision-maker lacks knowledge of the probability of being in the bad state or in the good state of nature, i.e. he does not know the value of the parameter ϵ.

Let us consider the smooth ambiguity model axiomatized by KMM (2005). According to this model, the decision-maker's welfare writes as

$$V(w, p+\tilde{\epsilon}) = E[\Phi\{(1-(p+\tilde{\epsilon}))u^G(w) + (p+\tilde{\epsilon})u^B(w)\}] \tag{2}$$

where E denotes the expectation operator over the random variable $\tilde{\epsilon}$, which probability distribution is assumed to be implicitly known. The function Φ captures the attitude towards ambiguity and is supposed to be smooth and increasing, i.e. $\Phi' > 0$. The decision-maker is considered as strictly ambiguity-averse if and only if Φ is strictly concave, as shown by KMM (2005). $\Phi'' < 0$ represents then strict ambiguity aversion, and $\Phi(x) = x$ represents ambiguity neutrality.

In the same way as in the expected utility model, where the addition of $\tilde{\epsilon}$ on the wealth level w reduces the utility of a risk-averse decision-maker[2], in the ambiguity model of KMM (2005), the introduction of $\tilde{\epsilon}$ on the probability p reduces the decision-maker's welfare if he is ambiguity-averse, compared to the case where the probability does not face any ambiguity. Indeed, $\Phi'' < 0$ implies

$$E[\Phi\{(1-(p+\tilde{\epsilon}))u^G(w) + (p+\tilde{\epsilon})u^B(w)\}] < \Phi(E[\{(1-(p+\tilde{\epsilon}))u^G(w) + (p+\tilde{\epsilon})u^B(w)\}]),$$

which can be rewritten equivalently (using our notations),

$$V(w, p+\tilde{\epsilon}) < \Phi(V_0(w, p+E(\tilde{\epsilon}))). \tag{3}$$

Note that for an ambiguity-neutral decision-maker, the presence of $\tilde{\epsilon}$ does not modify the welfare. Indeed, when $\Phi(x) = x$, we have:

$$E[\Phi\{(1-(p+\tilde{\epsilon}))u^G(w) + (p+\tilde{\epsilon})u^B(w)\}] = \Phi(E[\{(1-(p+\tilde{\epsilon}))u^G(w) + (p+\tilde{\epsilon})u^B(w)\}]),$$

that is equivalent to

$$V(w, p+\tilde{\epsilon}) = \Phi(V_0(w, p+E(\tilde{\epsilon}))) = V_0(w, p+E(\tilde{\epsilon})). \tag{4}$$

[2]Because risk aversion means that $E[u(w+\tilde{\epsilon})] < u(w+E(\tilde{\epsilon}))$.

While ambiguity aversion expresses preferences for a non-ambiguous event over an ambiguous event, it does not allow to express preferences over two ambiguous events. A more general issue is to define preferences between two ambiguous events. This is addressed in the next section.

3 Defining more ambiguity

To define one event as being more ambiguous than another, we consider the framework of Jewitt and Mukerji (2014) who provide conditions under which one act is more ambiguous than another, an act being a choice with contingent consequences.

Let's consider that the preferences of the decision-maker with a wealth level w are defined over a set of Anscombe-Aumann acts. Uncertainty is modelled by a state space Z containing all possible states of the world. For each state of the world ζ in Z, an act assigns lotteries to states, i.e. for each state ζ, the decision-maker welfare level will be $u^B(w)$ with probability $p + \epsilon_\zeta$, and will be $u^G(w)$ with probability $1 - (p + \epsilon_\zeta)$. An act assigns, for a given wealth level w, a distribution for the parameter ϵ_ζ, or equivalently, assigns a random variable $\tilde{\epsilon}$. An act, therefore, captures a level of ambiguity.

Following Jewitt and Mukerji (2014), one act is more ambiguous than another if an ambiguity-averse decision-maker prefers the first act to the second but an ambiguity-neutral is indifferent between the two acts[3]. Jewitt and Mukerji (2014)'s definition in the KMM model corresponds also to the definition of "an increase in ambiguity according to Ekern" proposed by Huang et al. (2015). In order to rewrite this definition of more ambiguity in our framework, let us consider a strictly ambiguity-averse decision-maker and define two acts, i.e. two random variables capturing ambiguity $\tilde{\epsilon}_1$ and $\tilde{\epsilon}_2$, with $E(\tilde{\epsilon}_1) = E(\tilde{\epsilon}_2)$ and for a given wealth level w. The decision-maker's utility writes now as

$$V(w, p + \tilde{\epsilon}_i) = E[\Phi\{(1 - (p + \tilde{\epsilon}_i))u^G(w) + (p + \tilde{\epsilon}_i)u^B(w)\}], \quad (5)$$

where ambiguity is captured by $\tilde{\epsilon}_i,\ i = 1, 2$.

Consequently, following Jewitt and Mukerji (2014), $\tilde{\epsilon}_2$ is more ambiguous than $\tilde{\epsilon}_1$ if an ambiguity-averse decision-maker (with Φ that satisfies $\Phi' > 0$ and $\Phi'' < 0$) prefers $\tilde{\epsilon}_1$ to $\tilde{\epsilon}_2$, i.e. if $V(w, p + \tilde{\epsilon}_2) \leq V(w, p + \tilde{\epsilon}_1)$. Note that an ambiguity-neutral decision-maker is indifferent between $\tilde{\epsilon}_1$ and $\tilde{\epsilon}_2$ since $E(\tilde{\epsilon}_1) = E(\tilde{\epsilon}_2)$.

An important feature of such a definition of more ambiguity is that the link between the random variables $\tilde{\epsilon}_1$ and $\tilde{\epsilon}_2$ is restricted to a statistical property of order 2 only. However, if the statistical link between $\tilde{\epsilon}_1$ and $\tilde{\epsilon}_2$ is of higher order than two, the comparison between $V(w, p+\tilde{\epsilon}_1)$ and $V(w, p+\tilde{\epsilon}_2)$ requires conditions on the sign of derivatives of Φ higher than

[3] More precisely, this definition corresponds to the definition (I) proposed by the authors. Jewitt and Mukerji (2014) also propose a second definition based on choices of agents with different levels of ambiguity aversion.

two. The aim of this article is to investigate how at higher orders the sign of the derivatives of Φ and the statistical link between $\tilde{\epsilon}_1$ and $\tilde{\epsilon}_2$ interacts to have $V(w, p+\tilde{\epsilon}_2) \leq V(w, p+\tilde{\epsilon}_1)$.

To do so, we assume that Φ is such that $(-1)^{(k+1)}\Phi^{(k)} > 0$ for all $k = 1, \ldots, n$, i.e. that the signs of the successive derivatives of the function Φ alternate. This is a natural assumption in the literature dealing with the KMM (2005) model. Indeed, in their original paper, KMM (2005) suggest using the function $\Phi(x) = \frac{-exp^{(-\alpha x)}}{\alpha}$ with $\alpha > 0$ (which exhibits constant absolute ambiguity aversion) as an illustration of their model (see also Taboga (2005), Collard et al. (2009)). It is easy to verify that this function is such that $(-1)^{(k+1)}\Phi^k > 0\ \forall k = 1, \ldots, n$, i.e. sharing the same properties as those exhibited above. This is also the case for the function $\Phi(x) = ln(x)$ for $x > 0$, or the function $\Phi(x) = \dfrac{x^\gamma}{\gamma}$ with $0 < \gamma < 1$ which are used by Gollier (2011) and Ju and Miao (2012) to express various attitudes towards ambiguity. As noted by Brockett and Golden (1987), all commonly used functions in economic theory, with the first derivative being positive and the second one being negative, have all successive derivatives that alternate in signs. Such functions are referred to as mixed risk-averse utility functions in the expected utility theory (see Caballé and Pomansky, 1996). By analogy, we call functions such as $(-1)^{(k+1)}\Phi^k > 0\ \forall k = 1, \ldots, n$ mixed ambiguity-averse functions.

To model changes in ambiguity between $\tilde{\epsilon}_1$ and $\tilde{\epsilon}_2$, we use the specific concept of stochastic dominance of order n defined by Ekern (1980) that establishes a partial ordering of probability distributions. Consider two random variables $\widetilde{X}$ and $\widetilde{Y}$ valued in some interval $[z_1, z_2]$ of the real line, with respective distribution functions F and G. Starting from $F_1 = F$ and $G_1 = G$, define iteratively for $z \in [z_1, z_2]$

$$F_{k+1}(z) = \int_{z_1}^{z} F_k(t)dt \text{ and } G_{k+1}(z) = \int_{z_1}^{z} G_k(t)dt$$

for $k \geq 1$. Then, $\widetilde{X}$ is said to be dominated by $\widetilde{Y}$ ($\widetilde{X} \preceq_n \widetilde{Y}$) via Ekern nth-order stochastic dominance if $G_n(z) \leq F_n(z)$ for all z where the inequality is strict for some z, and $E(\widetilde{X}^k) = E(\widetilde{Y}^k)\ \ \forall k = 1, \ldots, n-1$, i.e. the first $n-1$ moments being identical.

The concept of Ekern's dominance is very usual in risk theory. As an example, the notion of "mean-preserving increase in risk" introduced by Rothshild and Stiglitz (1970) is equivalent to Ekern's dominance of order 2 ($\widetilde{X} \preceq_2 \widetilde{Y}$). Similarly, the notion of "increase in downside risk" introduced by Menezes et al. (1980) is equivalent to Ekern's dominance of order 3 ($\widetilde{X} \preceq_3 \widetilde{Y}$).

We then have the following definition.

Definition

Let two random variables capturing two different levels of ambiguity, $\tilde{\epsilon}_1$ and $\tilde{\epsilon}_2$ be linked by an Ekern's dominance relation of order n. Given a strictly ambiguity-averse decision-maker with a function capturing ambiguity attitude Φ such that $(-1)^{(k+1)}\Phi^{(k)} > 0$ for *all $k, k = 1 \ldots n$, the random variable $\tilde{\epsilon}_2$ is considered by the decision-maker to be "more ambiguous" than the random variable $\tilde{\epsilon}_1$ if $V(w, p + \tilde{\epsilon}_2) \leq V(w, p + \tilde{\epsilon}_1)$.*

The next section proves that, contrary to intuition, a more ambiguous random variable is not always equivalent to a dominated variable in Ekern's sense, but depends on whether the order of the Ekern's dominance relation is even or odd.

4 Ambiguity apportionment

From Ekern (1980) and Ingersoll (1987), the following properties are well-known: $X \preceq_n Y$ is equivalent to $E[f(X)] \leq E[f(Y)]$ for all functions f such that $(-1)^{n+1} f^{(n)} > 0$, where $f^{(k)}$ denotes the k^{th} derivative of the function f.

Using these properties, we obtain the following proposition (see the proof in appendix 1).

Proposition 1

Let two random variables capturing two different levels of ambiguity, $\tilde{\epsilon}_1$ and $\tilde{\epsilon}_2$ be linked by an Ekern's dominance relation of order n. Given a strictly ambiguity-averse decision-maker with a function capturing ambiguity attitude Φ such that $(-1)^{(k+1)} \Phi^{(k)} > 0$ for all $k, k = 1 \ldots n$, the decision-maker considers the random variable $\tilde{\epsilon}_2$ to be more ambiguous than $\tilde{\epsilon}_1$ if $\tilde{\epsilon}_2 \preceq_n \tilde{\epsilon}_1$ when n is even, and if $\tilde{\epsilon}_1 \preceq_n \tilde{\epsilon}_2$ when n is odd.

Proposition 1 shows that, for a mixed ambiguity-averse decision-maker, a greater level of ambiguity between $\tilde{\epsilon}_2$ and $\tilde{\epsilon}_1$ is not always equivalent to the variable $\tilde{\epsilon}_2$ being dominated by $\tilde{\epsilon}_1$ in Ekern's sense[4], but can also correspond to the other way round[5].

The properties of the function Φ can be interpreted in terms of preferences for harms disaggregation, or preferences to "combine good with bad" in a similar way as the properties developed by Eeckhoudt and Schlesinger (2006) and Eeckhoudt et al. (2009) in the expected utility theory to explain the meaning of the alternation of signs of the successive derivatives of the utility function, i.e. the concept of risk apportionment of order n. We coin the term "ambiguity apportionment" of order n such preferences (see also Baillon, 2015) that coincide with preferences over simple random variables capturing the attitude towards ambiguity as presented in Proposition 1.

[4] We cannot extend this result to the case of a link of stochastic dominance between $\tilde{\epsilon}_1$ and $\tilde{\epsilon}_2$ since successive derivatives of the function f as defined in appendix 1 do not alternate in signs (see appendix 1).

[5] If ambiguity were defined on the good state of nature, contrary to what is usually done in the literature, a greater level of ambiguity would be equivalent to a dominated variable in Ekern's sense.

Let us recall the concept of risk apportionment. In the expected utility framework, Eeckhoudt et al. (2009) show that a decision-maker with a utility function u verifying $(-1)^{(N+M+1)}u^{(N+M)} > 0$ prefers the lottery $[w + X_N + Y_M, w + Y_N + X_M; \frac{1}{2}, \frac{1}{2}]$ to the lottery $[w + X_N + X_M, w + Y_N + Y_M; \frac{1}{2}, \frac{1}{2}]$ where random variables X_N, X_M, Y_N, Y_M are mutually independent and where X_k dominates Y_k to the order $k = N, M$ in Ekern's sense ($Y_k \preceq_k X_k$). The idea is that a decision-maker exhibiting $(N + M)$th-order risk apportionment preference will allocate the random variables in such a way as not to group the two "bad" random variables in the same state, where "bad" is defined via nth-order Ekern's dominance. The decision-maker prefers to combine "good" with "bad" and "bad" with "good" rather than "good" with "good" and "bad" with "bad". The random variable X_k is "good" and Y_k is "bad" because Y_k is dominated by X_k, i.e. Y_k is more risky than X_k. In risk theory, "more risky" coincides with "dominated in Ekern's sense"[6]. While in risk theory, harms are random variables added to the wealth level w, in our framework harms are random variables added to the probability level associated to the bad state of nature p. Contrary to intuition, Proposition 1 shows that "more ambiguous" does not always coincide with "dominated in Ekern's sense". The explanation of Proposition 1 is the following.

If $\tilde{\epsilon}_1$ and $\tilde{\epsilon}_2$ are degenerated random variables, $\tilde{\epsilon}_1 = 0$ and $\tilde{\epsilon}_2 = k$ with $k > 0$, then according to the definition of Ekern, $\tilde{\epsilon}_2$ dominates $\tilde{\epsilon}_1$ to the order of 1. However, the passage from $\tilde{\epsilon}_1$ to $\tilde{\epsilon}_2$ corresponds in our model to an increase in the probability of loss. Thus this passage is considered as an adverse or "bad" outcome for the decision-maker, and any individual such as $\Phi' > 0$ dislikes this. This explains why the passage from $\tilde{\epsilon}_1$ to $\tilde{\epsilon}_2$ constitutes an increase in ambiguity, where more ambiguity corresponds to a higher probability of loss.

If $\tilde{\epsilon}_1 = 0$ and $\tilde{\epsilon}_2 = \tilde{\epsilon}$ with $E(\tilde{\epsilon}) = 0$, then $\tilde{\epsilon}_1$ dominates $\tilde{\epsilon}_2$ to the order of 2 in Ekern's sense. The passage from $\tilde{\epsilon}_1$ to $\tilde{\epsilon}_2$ is a mean-preserving spread in the space of probabilities which is disliked by all individuals averse to ambiguity, i.e. such that $\Phi'' < 0$. Consequently, this passage is an adverse or "bad" outcome for the decision-maker, which explains why $\tilde{\epsilon}_2$ is considered as more ambiguous than $\tilde{\epsilon}_1$ by the decision-maker, where more ambiguity corresponds to an uncertain probability of loss[7]. This case is equivalent to the definition of increased ambiguity as proposed by Snow (2010), Jewitt and Mukerji (2014) and Huang et al. (2015).

Let us now consider the case $n = 3$ with the two following random variables: $\tilde{\epsilon}_1 = [k, \tilde{\epsilon}; \frac{1}{2}, \frac{1}{2}]$ and $\tilde{\epsilon}_2 = [k + \tilde{\epsilon}, 0; \frac{1}{2}, \frac{1}{2}]$. Using the property of Ingersoll (1987), it is easy to

[6] Eeckhoudt et al. (2009) results also apply to stochastic dominance.

[7] As suggested by one referee, for $n = 1$ and $n = 2$, changes in ambiguity can be represented through the linear transformation function proposed by Sandmo (1971) $t(p) = \gamma(p - \overline{p}) + \overline{p} + k$, where γ is a multiplicative shift parameter, k is an additive shift parameter, and $\overline{p}$ is the mean probability. For $n = 1$, an increase in k (from $k = 0$ with $\gamma = 1$) is similar to a first-degree risk improvement. For $n = 2$, an increase in γ (from $\gamma = 1$ with $k = 0$) is similar to a Rothschild-Stiglitz increase in risk.

verify that $\tilde{\epsilon}_2$ dominates $\tilde{\epsilon}_1$ in Ekern's sense to the order of 3. However, Proposition 1 means that an ambiguity-prudent decision-maker, i.e. such as $\Phi^{(3)}(x) > 0$ following the terminology of Baillon (2015), prefers $\tilde{\epsilon}_1$ to $\tilde{\epsilon}_2$. Why is $\tilde{\epsilon}_2$ considered as more ambiguous than $\tilde{\epsilon}_1$? The intuitive explanation is the following. Recall that k and $\tilde{\epsilon}$ represent adverse or "bad" outcomes for the decision-maker. Consequently, an ambiguity-prudent decision-maker prefers not to be confronted with these two adverse outcomes together in one state of nature as is the case with the lottery $\tilde{\epsilon}_2$. He prefers rather to disaggregate these two adverse outcomes across states of nature as is the case with the lottery $\tilde{\epsilon}_1$. Hence, a more ambiguous random variable defined through a specific case of stochastic dominance of order 3 makes an ambiguity-averse worse off only if he is also ambiguity-prudent. This case is equivalent to the definition of "increased in downside ambiguity according to Ekern" introduced by Huang et al. (2015).

Such preference for disaggregation of adverse outcomes also applies for higher orders. Indeed, let us consider now the case $n = 4$ with the following random variables: $\tilde{\epsilon}_1 = [\tilde{\theta}, \tilde{\epsilon}; \frac{1}{2}, \frac{1}{2}]$ and $\tilde{\epsilon}_2 = [\tilde{\theta} + \tilde{\epsilon}, 0; \frac{1}{2}, \frac{1}{2}]$, with $E(\tilde{\theta}) = 0$ and $\tilde{\theta}$ and $\tilde{\epsilon}$ are independent. According to the definition of Ekern, $\tilde{\epsilon}_2$ is dominated by $\tilde{\epsilon}_1$ to the order of 4. Proposition 1 tells us that $\tilde{\epsilon}_2$ is considered to be more ambiguous than $\tilde{\epsilon}_1$. Indeed, both risks $\tilde{\epsilon}$ and $\tilde{\theta}$ are adverse outcomes for the individual. An ambiguity-temperant decision-maker, i.e. such as $\Phi^{(4)}(x) < 0$ following the terminology of Baillon (2015), prefers to disaggregate these adverse outcomes rather than aggregate them, and thus prefers $\tilde{\epsilon}_1$ to $\tilde{\epsilon}_2$. The random variable $\tilde{\epsilon}_1$ is thus considered as less ambiguous than $\tilde{\epsilon}_2$.

To summarize and using the notations of Eeckhoudt et al. (2009), the case $n = 3$ corresponds to $N = 1$ and $M = 2$ with "bad" $\equiv Y_N = k$ and "good" $\equiv X_N = 0$, "bad" $\equiv Y_M = \tilde{\epsilon}$ and "good" $\equiv X_M = 0$. The case $n = 4$ corresponds to $N = 2 = M$ with "bad" $\equiv Y_N = \tilde{\theta}$ (with $\tilde{\theta}$ and $\tilde{\epsilon}$ independent), and "good" $\equiv X_N = 0$, "bad" $\equiv Y_M = \tilde{\epsilon}$ and "good" $\equiv X_M = 0$. For higher orders, all even orders n can be written as $n = N + M$ with $N = 2$ and M even, and where "bad" $\equiv Y_N = \tilde{\theta}$ and "good" $\equiv X_N = 0$, and "bad" $\equiv Y_M$ and "good" $\equiv X_M$ for all X_M and Y_M such that $Y_M \preceq_M X_M$. When n is even, $\tilde{\epsilon}_2$ can be written as the lottery $[w + X_N + X_M, w + Y_N + Y_M; \frac{1}{2}, \frac{1}{2}]$, and $\tilde{\epsilon}_1$ as the lottery $[w + X_N + Y_M, w + Y_N + X_M; \frac{1}{2}, \frac{1}{2}]$. It is easy to show that $\tilde{\epsilon}_2 \preceq_{N+M} \tilde{\epsilon}_1$. The decision-maker prefers combining "good" with "bad" and then prefers $\tilde{\epsilon}_1$ that dominates $\tilde{\epsilon}_2$. For all odd orders n, n can be written as $n = N + M$ with $N = 1$ and M even where "bad" $\equiv Y_N = k$ and "good" $\equiv X_N = 0$, "bad" $\equiv Y_M$ and "good" $\equiv X_M$ for all X_M and Y_M such that $Y_M \preceq_M X_M$ with M even. When n is odd, $\tilde{\epsilon}_2$ can be written as the lottery $[w + X_N + X_M, w + Y_N + Y_M; \frac{1}{2}, \frac{1}{2}]$ and $\tilde{\epsilon}_1$ as $[w + X_N + Y_M, w + Y_N + X_M; \frac{1}{2}, \frac{1}{2}]$. It is easy to show that in this case, $\tilde{\epsilon}_1$ is dominated by $\tilde{\epsilon}_2$ ($\tilde{\epsilon}_1 \preceq_{N+M} \tilde{\epsilon}_2$). But as previously explained, the decision-maker prefers combining "good" with "bad" rather than "good" with "good" and "bad" with "bad" and then prefers $\tilde{\epsilon}_1$ to $\tilde{\epsilon}_2$ despite the fact that $\tilde{\epsilon}_1$ is dominated by $\tilde{\epsilon}_2$.

This interpretation is similar to the one developed in expected utility theory. These higher-order ambiguity attitudes entail a preference for combining relatively good outcomes with bad ones and can be interpreted as a desire to disaggregate the harms of unavoidable random variables and losses[8].

5 Conclusion

While ambiguity aversion expresses preferences for non-ambiguous situations over ambiguous situations, this paper goes one step further and proposes preferences over more ambiguous probabilities in the specific case of two states of nature. Changes in ambiguity over probability distributions are expressed through the specific notion of stochastic dominance of order n defined by Ekern (1980). A random variable being more ambiguous than another for the individual is equivalent to having the first variable dominated to the order n by the second one in Ekern's sense when n is even and the second variable being dominated to the order n by the first one in Ekern's sense when n is odd. This notion of changes in ambiguity is linked to the signs of the successive derivatives of the function capturing the individual attitudes towards ambiguity. These properties, referred to as ambiguity apportionment, are interpreted in terms of preferences for harms disaggregation over probabilities.

Our results imply that, contrary to previous literature, changes in ambiguity are not defined only in relation to ambiguity aversion or ambiguity prudence. Hence, further conditions on ambiguity attitudes of higher orders are required to have more ambiguity making a more ambiguity-averse individual worse off.

There are limitations of this analysis that need to be pointed out. First, we have considered the model of KMM (2005) to express ambiguity preferences. An extension of this paper would be to consider other models of decisions incorporating attitude towards ambiguity. Yet, it should be stressed that the KMM (2005) model is especially adapted to our analysis since individuals' subjective beliefs about objective probabilities are represented by a probability distribution which makes it easy to use the specific concept of stochastic dominance of order n. Second, we have considered a binary risk where ambiguity impacts only two states of nature. A generalization would be to consider ambiguity impacting a continuum state of nature. Extending our work in these directions would provide some interesting topics for future research.

[8] In Eeckhoudt and Schlesinger (2006) and Eeckhoudt et al. (2009), lotteries are independent 50-50 lotteries. A more general case is to consider dependent and binary lotteries which are not necessarily 50-50 lotteries (see Denuit and Rey, 2013).

Appendix 1

$V(w, p+\tilde{\epsilon}_2) \leq V(w, p+\tilde{\epsilon}_1)$ is equivalent to

$E[\Phi\{(1-(p+\tilde{\epsilon}_2))u^G(w)+(p+\tilde{\epsilon}_2)u^B(w)\}] \leq E[\Phi\{(1-(p+\tilde{\epsilon}_1))u^G(w)+(p+\tilde{\epsilon}_1)u^B(w)\}]$, that rewrites as:

$E[\Phi\{V_0(w,p)+\tilde{\epsilon}_2\Delta u(w)\}] \leq E[\Phi\{V_0(w,p)+\tilde{\epsilon}_1\Delta u(w)\}]$, with $\Delta u(w) = u^B(w)-u^G(w)$.

Let us define the function f as follows: $f(\epsilon) = \Phi\Big(V_0(w,p) + \epsilon\Delta u(w)\Big)$.

The previous inequality rewrites as: $E[f(\widetilde{\epsilon}_2)] \leq E[f(\widetilde{\epsilon}_1)]$.

We obtain:

$f'(\epsilon) = \Delta u(w)\Phi'\Big(V_0(w,p) + \epsilon(\Delta u(w))\Big)$,

$f''(\epsilon) = (\Delta u(w))^2\Phi''\Big(V_0(w,p) + \epsilon(\Delta u(w))\Big)$,

$f^{(3)}(\epsilon) = (\Delta u(w))^3\Phi^{(3)}\Big(V_0(w,p) + \epsilon(\Delta u(w))\Big)$,

$f^{(4)}(\epsilon) = (\Delta u(w))^4\Phi^{(4)}\Big(V_0(w,p) + \epsilon(\Delta u(w))\Big)$,

... ,

$f^{(n)}(\epsilon) = (\Delta u(w))^n\Phi^{(n)}\Big(V_0(w,p) + \epsilon(\Delta u(w))\Big)$.

As by assumption, $\Delta u(w) < 0$, we obtain:

$\Phi'(x) > 0 \ \forall x \Leftrightarrow f'(\epsilon) < 0 \ \forall\epsilon$,

$\Phi''(x) < 0 \ \forall x \Leftrightarrow f''(\epsilon) < 0 \ \forall\epsilon$,

$\Phi^{(3)}(x) > 0 \ \forall x \Leftrightarrow f^{(3)}(\epsilon) < 0 \ \forall\epsilon$,

$\Phi^{(4)}(x) < 0 \ \forall x \Leftrightarrow f^{(4)}(\epsilon) < 0 \ \forall\epsilon$,

$\ldots$,

$\Phi^{(n)}(x) > 0 \ \forall x$ when n is odd $\Leftrightarrow f^{(n)}(\epsilon) < 0 \ \forall\epsilon$,

$\Phi^{(n)}(x) < 0 \ \forall x$ when n is even $\Leftrightarrow f^{(n)}(\epsilon) < 0 \ \forall\epsilon$.

Using Ekern (1980), we obtain:

if $\tilde{\epsilon}_2 \preceq_n \tilde{\epsilon}_1$ for n even, $E[f(\widetilde{\epsilon}_2)] \leq E[f(\widetilde{\epsilon}_1)]$ for all f such that $f^{(n)}(\epsilon) < 0 \ \forall\epsilon$,

if $\tilde{\epsilon}_1 \preceq_n \tilde{\epsilon}_2$ for n odd, $E[f(\widetilde{\epsilon}_2)] \leq E[f(\widetilde{\epsilon}_1)]$ for all f such that $f^{(n)}(\epsilon) < 0 \ \forall\epsilon$, that proves Proposition 1.

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