TAIL ASYMPTOTIC BEHAVIOR OF THE SUPREMUM OF A CLASS OF CHI-SQUARE PROCESSES

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Abstract: We analyze in this paper the supremum of a class of chi-square processes over non-compact intervals, which can be seen as a multivariate counterpart of the generalized weighted Kolmogorov-Smirnov statistic. The boundedness and the exact tail asymptotic behavior of the supremum are derived. As examples, the chi-square process generated from the Brownian bridge and the fractional Brownian motion are discussed.

Key Words: chi-square process; exact asymptotics; Brownian bridge

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1. Introduction

Let \( X(t), t \geq 0 \), be a Gaussian process with almost surely (a.s.) continuous sample paths. For a sequence of constants \( \{b_i\}_{i=1}^n \) satisfying

\[
1 = b_1 = \cdots = b_k > b_{k+1} \geq \cdots \geq b_n > 0
\]

we define the chi-square process as

\[
\chi_b^2(t) = \sum_{i=1}^n b_i^2 X_i^2(t), \quad t \geq 0,
\]

where \( X_i \)'s are independent copies of \( X \). The supremum of chi-square process appears naturally as limiting test statistic in various statistical models; see, e.g., [1, 2]. It also plays an important role in reliability applications in the engineering sciences, see [3, 4] and the references therein.

Of interest in applied probability and statistics is the tail asymptotics of \( \sup_{t \in T} \chi_b^2(t) \) for an interval \( T \subset \mathbb{R}_+ \), provided that

\[
\sup_{t \in T} \chi_b^2(t) < \infty \quad \text{a.s.}
\]

Numerous contributions have been devoted to the study of the tail asymptotics of the supremum of chi-square processes over compact intervals \( T \); see, e.g., [3, 5] and the references therein, where the technique used is to transform the supremum of chi-square process into the supremum of a special Gaussian random field. We refer to, e.g., [6, 7, 8, 9] for more discussions on the tail asymptotics (or excursion probability) of Gaussian and related fields.

In this paper, we are interested in the analysis of the supremum of a class of weighted locally stationary chi-square processes defined by

\[
\sup_{t \in T} \frac{\chi_b^2(t)}{w^2(t)}, \quad \text{with } T = (0, 1) \text{ or } (0, 1],
\]

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where \( w(\cdot) \) is some positive continuous function definable on the non-compact set \( \mathcal{T} \), and the generic process \( X \) is the locally stationary Gaussian process. More precisely, \( X(t), t \in \mathcal{T} \), is a centered Gaussian process with a.s. continuous sample paths, unit variance and correlation function \( r(\cdot, \cdot) \) satisfying

\[
\lim_{h \to 0} \frac{1 - r(t, t + h)}{K^2(|h|)} = C(t)
\]

uniformly in \( t \in I \), for all the compact interval \( I \) in \( \mathcal{T} \), where \( K(\cdot) \) is a positive regularly varying function at 0 with index \( \alpha/2 \in (0, 1] \), and \( C(\cdot) \) is a positive continuous function satisfying

\[
\lim_{t \to 0} C(t) = \infty \quad \text{or} \quad \lim_{t \to 1} C(t) = \infty.
\]

We refer to [10] for more discussions on such locally stationary Gaussian processes.

Our motivation for considering the supremum of the weighted locally stationary chi-square processes over the non-compact interval \( \mathcal{T} = (0, 1) \) or \( (0, 1] \) is from its potential applications in statistics. For instance, in its univariate framework (with \( n = 1 \)) the following generalized weighted Kolmogorov-Smirnov statistic

\[
W_w := \sup_{t \in (0, 1)} \frac{|B(t)|}{w(t)}, \quad \text{with} \quad B(t) = \frac{B(t)}{\sqrt{t(1-t)}}, \quad t \in (0, 1),
\]

has been discussed in the statistics literature, see, e.g., [11], where \( B \) is the standard Brownian bridge with variance function \( \text{Var}(B(t)) = t(1-t), t \in [0, 1] \) and \( w \) is a suitably chosen weight function such that

\[
W_w < \infty \quad \text{a.s.}
\]

We refer to [11, 12, 13, 14] for further discussions on the generalized weighted Kolmogorov-Smirnov statistic.

An interesting theoretical question is to find sufficient and necessary conditions on \( w \) so that the a.s. finiteness of (4) holds. It is shown in [11][Theorem 3.3, Theorem 4.2.3] (see also [12][Theorem 26.3]) that

\[
W_w < \infty \quad \text{a.s.} \quad \Leftrightarrow \quad \int_0^1 \frac{1}{t(1-t)} e^{-ctw^2(t)} dt < \infty \quad \text{for some} \quad c > 0.
\]

One of the main results displayed in Theorem 3.1 shows necessary and sufficient conditions on the weight function \( w \) under which it holds that

\[
\sup_{t \in \mathcal{T}} \frac{\chi^2(t)}{w^2(t)} < \infty \quad \text{a.s.}
\]

This extends the result of (5). Furthermore, for certain \( w \) satisfying (6) we derive in Theorem 3.3 the exact asymptotics of

\[
\mathbb{P} \left( \sup_{t \in \mathcal{T}} \frac{\chi^2(t)}{w^2(t)} > u \right), \quad u \to \infty.
\]

As an important application of Theorem 3.3, we obtain in Corollary 3.4 the tail asymptotics of the supremum of the chi-square process generated from the Brownian bridge. It is worth mentioning that this tail asymptotic result is new even for the univariate (i.e., \( n = 1 \)) case. As a second example, the chi-square process generated by the fractional Brownian motion is discussed.

We expect that the derived results will have interesting applications in statistics or beyond.

**Organization of the rest of the paper:** In Section 2 we present a preliminary result which is a tailored version of Theorem A.1 in [10]. The main results are given in Section 3, followed by examples. All the proofs are displayed in Section 4.
2. Preliminaries

This section concerns a result derived in [10], which is crucial for the derivation of (6). Based on the discussions therein, we shall consider \( \int_0^{1/2} (C(s))^{1/\alpha} \, ds = \infty \) or \( \int_{1/2}^{1} (C(s))^{1/\alpha} \, ds = \infty \). For this purpose, of crucial importance is the following function

\[
 f(t) = \int_{1/2}^{t} (C(s))^{1/\alpha} \, ds, \quad t \in (0,1).
\]

We denote by \( \int f(t), t \in (f(0), f(1)) \) the inverse function of \( f(t), t \in (0,1) \). Further, for any \( d > 0 \), let \( s_{j,d}^{(1)} = \int f(jd), j \in \mathbb{N} \cup \{0\} \) if \( f(1) = \infty \), and let \( s_{j,d}^{(0)} = \int f(-jd), j \in \mathbb{N} \cup \{0\} \) if \( f(0) = -\infty \). Denote \( \Delta_{j,d}^{(1)} = [s_{j,d}^{(1)}, s_{j,d}^{(1)}], j \in \mathbb{N} \) and \( \Delta_{j,d}^{(0)} = [s_{j,d}^{(0)}, s_{j,d}^{(0)}], j \in \mathbb{N} \), which give a partition of \([1/2, 1)\) in the case \( f(1) = \infty \) and a partition of \((0,1/2)\) in the case \( f(0) = -\infty \), respectively. Moreover, let \( q(u) = \int K(u^{-1/2}) \) be the inverse function of \( K(\cdot) \) at point \( u^{-1/2} \) (assumed to exist asymptotically).

The following (scenario-dependent) restrictions on the positive continuous weight function \( w^2 \) and the correlation function \( r(\cdot, \cdot) \) of \( X \) play a crucial role. Let therefore \( S \in \{0,1\} \).

**Condition A(S):** The weight function \( w^2 \) is monotone in a neighbourhood of \( S \) and satisfies \( \lim_{t \to S} w^2(t) = \infty \).

**Condition B(S):** Suppose that there exists some constant \( d_0 > 0 \) such that

\[
 \limsup_{j \to \infty} \sup_{t \neq s \in \Delta_{j,d_0}^{(s)}} \frac{1 - r(t,s)}{K^2(|f(t) - f(s)|)} < \infty,
\]

and when \( \alpha = 2 \) and \( k = 1 \), assume further

\[
 K^2(|t|) = O(t^2), \quad t \to 0.
\]

**Condition C(S):** Suppose that there exists some constant \( d_0 > 0 \) such that

\[
 \liminf_{j \to \infty} \inf_{t \neq s \in \Delta_{j,d_0}^{(s)}} \frac{1 - r(t,s)}{K^2(|f(t) - f(s)|)} > 0.
\]

Moreover, there exist \( j_0, l_0 \in \mathbb{N}, M_0, \beta > 0 \), such that for \( j \geq j_0, l \geq l_0 \),

\[
 (8) \quad \sup_{s \in \Delta_{j+l,d_0}^{(s)}, t \in \Delta_{j,d_0}^{(s)}} |r(s,t)| < M_0 l^{-\beta}.
\]

For the subsequent discussions we present a tailored version of Theorem A.1 of [10], focusing on \( |f(S)| = \infty \). We define

\[
 I_w(S) = \left| \int_{1/2}^{S} (C(t))^{1/\alpha} \frac{w(t)^{k-2}}{\bar{q}(w^2(t))} e^{-w^2(t)/2} \, dt \right|.
\]

**Theorem 2.1.** Let \( X(t), t \in (0,1) \), be a centered locally stationary Gaussian process with a.s. continuous sample paths, unit variance and correlation function \( r(\cdot, \cdot) \) satisfying (3) and \( r(s,t) < 1 \) for \( s \neq t \in (0,1) \). Suppose further that, for \( S = 0 \) or 1, we have \( |f(S)| = \infty \) and A(S), B(S), C(S) are satisfied. Then

\[
 (9) \quad \mathbb{P} \{ \chi^2_{B}(t) \leq w^2(t) \text{ ultimately as } t \to S \} = 0, \quad \text{or} \quad 1
\]

according to

\[
 I_w(S) = \infty, \quad \text{or} \quad < \infty.
\]
3. Main Results

In this section, we first give a criteria for (6) to hold and then display the exact asymptotics of (7) for different types of $w$ such that (6) is valid.

3.1. Analysis of (6). Denote by $E(0) = (0, 1/2]$ and $E(1) = [1/2, 1)$. Recall that $S \in \{0, 1\}$. Under the conditions of Theorem 2.1, we have that if $I_w(S) < \infty$, then

$$\sup_{t \in E(S)} \frac{\chi^2_b(t)}{w^2(t)} < \infty \quad \text{a.s.},$$

however, when $I_w(S) = \infty$ we only see that

$$\sup_{t \in E(S)} \frac{\chi^2_b(t)}{w^2(t)} \geq 1 \quad \text{a.s.}$$

Apparently, the above is not informative for the claim in (6). On the other hand, it is easily shown that

$$\sup_{t \in E(S)} \frac{\chi^2_b(t)}{w^2(t)} < \infty \quad \text{a.s.} \iff \sup_{t \in E(S)} \frac{|X(t)|}{w(t)} < \infty \quad \text{a.s.},$$

which means that, instead of the condition $I_w(S) = \infty$ in Theorem 2.1, a more accurate condition that is independent of $n, k$ should be possible to ensure that (6) holds. Inspired by this fact, we provide below a sufficient and necessary condition for (6) to hold.

Define, for any constant $c > 0$ and any positive continuous function $w$

$$J_{c,w}(S) = \left| \int_{1/2}^{S} (C(t))^{1/\alpha} e^{-cw^2(t)} dt \right|.$$

Below is our first principal result, a criterion for (6), which is a generalization of (5).

**Theorem 3.1.** Under the conditions of Theorem 2.1 we have

$$\sup_{t \in E(S)} \frac{\chi^2_b(t)}{w^2(t)} < \infty \quad \text{a.s.} \iff J_{c,w}(S) < \infty \quad \text{for some} \quad c > 0.$$

Next we illustrate the criteria presented in Theorem 3.1 by an example of a weighted chi-square process with generic process being the normalized standard Brownian bridge, which further provides us with a clear comparison between $I_w(S)$ and $J_{c,w}(S)$.

**Example 3.2.** Let $X(t) = \overline{B}(t), t \in (0, 1)$, and, with $\rho_1 > 0, \rho_2 \in \mathbb{R}$, define

$$w^{\rho_1, \rho_2}_2(t) = 2\rho_1 \ln \ln \left( \frac{e^2}{t(1-t)} \right) + 2\rho_2 \ln \ln \ln \left( \frac{e^2}{t(1-t)} \right), \quad t \in (0, 1).$$

First note that for the normalized standard Brownian bridge

$$\lim_{h \to 0} \frac{1 - \mathbb{E} \left( \overline{B}(t) \overline{B}(t+h) \right)}{|h|} = \frac{1}{2t(1-t)}$$

holds uniformly in $t \in I$, for any compact interval $I$ in $(0, 1)$. This means that $\overline{B}$ is a locally stationary Gaussian process with

$$K(h) = |h|, \quad \alpha = 1, \quad q(u) = u^{-1}.$$

Furthermore,

$$f(t) = \int_{1/2}^{t} \frac{1}{2s(1-s)} ds = \frac{1}{2} \ln \left( \frac{t}{1-t} \right)$$
implying that $f(1) = -f(0) = \infty$. Moreover, by the proof of Corollary 2.6 in [10] we have that conditions $B(S)$ and $C(S)$ are satisfied by $B(t), t \in (0, 1)$, and $E(B(t), B(s)) < 1$ for $s \neq t, s, t \in (0, 1)$. Thus, all the conditions of Theorem 3.1 are fulfilled.

Next, on one hand, we have

$$
\frac{1}{t(1-t)} (w_{*,2}(t))^k e^{-\frac{w_{*,2}(t)}{2}} \sim \frac{Q}{t(1-t)} \left( \ln \left( \frac{1}{t(1-t)} \right) \right)^{\rho_1} \left( \ln \ln \left( \frac{1}{t(1-t)} \right) \right)^{\rho_2-k/2}
$$

as $t \to 0$ or $t \to 1$, with $Q$ some positive constant. Thus, elementary calculations show that

$$
I_w(0) = I_w(1) = \int_{1/2}^{1} \frac{(w_{*,2}(t))^k}{t(1-t)} e^{-\frac{w_{*,2}(t)}{2}} dt < \infty
$$

holds if and only if

$$
(12) \quad \rho_1 > 1, \quad \text{or} \quad \rho_1 = 1 \text{ and } \rho_2 > 1 + k/2.
$$

On the other hand, we can show that the functions $w_{*,2}(t)$ satisfying that $\exists c > 0$ such that $J_{c,w}(S) < \infty$ are not restricted to the ones satisfying (12). In fact, since for any $\rho_1 > 0$ there exists some $c$ such that $\rho_1 > \frac{1}{2c}$, we have that

$$
J_{c,w}(0) = J_{c,w}(1) = \int_{1/2}^{1} \frac{1}{t(1-t)} e^{-cw_{*,2}(t)} dt
$$

$$
\leq \int_{1/2}^{1} \frac{1}{t(1-t)} \left( \ln \left( \frac{1}{t(1-t)} \right) \right)^{2c\rho_1} \left( \ln \ln \left( \frac{1}{t(1-t)} \right) \right)^{2c\rho_2} dt < \infty
$$

holds for any $\rho_2 \in \mathbb{R}$. Thus, we conclude from Theorem 3.1 that

$$
\sup_{t \in (0,1)} \frac{\chi^2(t)}{w_{*,2}(t)} < \infty \quad \text{a.s.}
$$

holds for any $\rho_1 > 0$ and $\rho_2 \in \mathbb{R}$.

### 3.2. Asymptotics of (7)

For those $w$ such that (6) holds, of interest is the exact tail asymptotic behavior of $\sup_{t \in \mathcal{T}} \frac{\chi^2(t)}{w(t)}$. Actually, as we have seen, the behavior of $w$ around 0 and 1 plays a crucial role for the finiteness in (6). However, this does not apply to the tail asymptotics of $\sup_{t \in \mathcal{T}} \frac{\chi^2(t)}{w(t)}$. It turns out that only the probability mass in the neighborhood of minimizer of $w$ contribute to the tail asymptotics. As discussed in [15], the weight function is introduced when constructing the Goodness-of-Fit test which is intended to emphasize a specific region of the domain. With these motivations, for the tail asymptotics we shall consider the following two types of $w$:

**Assumption F1:** The function $w$ attains its minimum at finite distinct inner points $\{t_i\}_{i=1}^m$ of $\mathcal{T}$, and

$$
(13) \quad w(t_i + t) = w(t_i) + a_i |t_i|^{\beta_i} (1 + o(1)), \quad t \to t_i
$$

holds for some positive constants $a_i, \beta_i > 0, i = 1, 2, \ldots, m$.

**Assumption F2:** The function $w$ attains its minimum at all points on disjoint intervals $[c_i, d_i] \subseteq \mathcal{T}, i = 1, 2, \ldots, m$ (i.e., $w$ is a constant on these intervals).

Under assumption F1, we need additional conditions which are stated below. Recall $q(u) = \hat{K}(u^{-1/2})$. It follows that $q(u)$ is a regularly varying function at infinity with index $-1/\alpha$ which can be further expressed as $q(u) = u^{-1/\alpha} L(u^{-1/2})$, with $L(\cdot)$ a slowly varying function at 0. Denote further $\beta = \max_{1 \leq i \leq m} \beta_i$. According to the values of $L(u^{-1/2})$ as $u \to \infty$, we consider the following three scenarios:

**C1(\beta):** $\beta > \alpha$, or $\beta = \alpha$ and $\lim_{u \to \infty} L(u^{-1/2}) = 0$;
\textbf{C2(\(\beta\))}: \(\beta = \alpha\) and \(\lim_{u \to \infty} L(u^{-1/2}) = L \in (0, \infty)\);

\textbf{C3(\(\beta\))}: \(\beta < \alpha\), or \(\beta = \alpha\) and \(\lim_{u \to \infty} L(u^{-1/2}) = \infty\).

Before displaying our results, we introduce two important constants. One is the \textit{Pickands constant} defined by

\[
\mathcal{H}_{2H} = \lim_{s \to \infty} \frac{1}{s} \mathbb{E}\left( \exp\left( \sup_{t \in [0,s]} \left( \sqrt{2B_H(t)} - t^{2H}\right) \right) \right),
\]

with \(B_H(t), t \in \mathbb{R}\), a standard fractional Brownian motion (fBm) defined on \(\mathbb{R}\) with Hurst index \(H \in (0,1]\). And the other one is the \textit{Piterbarg constant} defined by

\[
\mathcal{P}_{2H} = \lim_{\lambda \to \infty} \mathbb{E}\left( \exp\left( \sup_{t \in [-\lambda,\lambda]} \left( \sqrt{2B_H(t)} - (1 + d) |t|^{2H}\right) \right) \right), \quad d > 0.
\]

We refer to [16] for the properties and generalizations of the Pickands-Piterbarg type constants. In what follows, \(\alpha\) will play a similar role as \(2H\). Moreover, We shall use the standard notation for asymptotic equivalence of two functions \(f\) and \(h\). Specifically, we write \(f(x) \sim h(x)\), if \(\lim_{x \to a} f(x)/h(x) = 1\) (\(a \in \mathbb{R} \cup \{\infty}\)), and further, write \(f(x) = o(h(x))\), if \(\lim_{x \to a} f(x)/h(x) = 0\).

Let \(K = \{1 \leq i \leq m : \beta_i = \beta\}\) and \(K^c = \{1 \leq i \leq m : \beta_i < \beta\}\). Below is our second principal result.

\textbf{Theorem 3.3.} Let \(\frac{x^2(t)}{w^2(t)}, t \in T\), be the weighted locally stationary chi-square process considered in Theorem 2.1 such that (6) holds. We have:

(i). If \(\textbf{F1}\) is satisfied, then, as \(u \to \infty\),

\[
\mathbb{P}\left\{ \sup_{t \in T} \frac{x^2(t)}{w^2(t)} > u \right\} \sim \left( \prod_{i=k+1}^{n} (1 - b_i^2)^{-1/2} \right) \mathcal{M}(u) \mathcal{Y}_k(w^2(t_1)u),
\]

where (with the convention \(\prod_{i=p}^{q} = 1\) if \(q < p\))

\[
\mathcal{Y}_k(u) := \mathbb{P}\{\chi^2_{k,1}(0) > u\} = \frac{2(2-k)^2}{\Gamma(k/2)} u^{k/2-1} \exp\left( -\frac{u}{2} \right), \quad u > 0,
\]

and

\[
\mathcal{M}(u) = \begin{cases} 
2 \left( \sum_{i \in K} \alpha_i^{1-\beta} (C(t_i))^{1/\alpha} \right) (w(t_1))^{2/\alpha - 1/\beta} \Gamma(1/\beta + 1) H_{\alpha}(q(u))^{-1} u^{-1/\beta}, & \text{for C1(\(\beta\))}, \\
\sum_{i \in K} \mathcal{P}_{\alpha_i(w(t_i)C(t_i))^{-1}}^{a_i} + \mathbb{E}K^c, & \text{for C2(\(\beta\))}, \\
m, & \text{for C3(\(\beta\))}.
\end{cases}
\]

(ii). If \(\textbf{F2}\) is satisfied, then, as \(u \to \infty\),

\[
\mathbb{P}\left\{ \sup_{t \in T} \frac{x^2(t)}{w^2(t)} > u \right\} \sim \left( \prod_{i=k+1}^{n} (1 - b_i^2)^{-1/2} \right) \left( \sum_{j=1}^{m} \int_{t_{j-1}}^{t_j} (C(t))^{1/\alpha} dt \right) \mathcal{H}_{\alpha} (q(w^2(c_1)u))^{-1} \mathcal{Y}_k(w^2(c_1)u).
\]

We conclude this section with two applications of Theorem 3.3. Full proofs of Corollaries 3.4 and 3.5 can be found in [17].

\textbf{Corollary 3.4}. Let \(\frac{x^2(t)}{w^2(t)} t \in (0,1), \) with \(\rho_1 > 0\) and \(\rho_2 \in \mathbb{R}\), be the weighted locally stationary chi-square process discussed in Example 3.2. We have, as \(u \to \infty\), if \(\rho_2 \geq -\rho_1 \ln \ln(4e^2)\), then

\[
\mathbb{P}\left\{ \sup_{t \in (0,1)} \frac{x^2(t)}{w^2(t)} > u \right\} \sim \left( \prod_{i=k+1}^{n} (1 - b_i^2)^{-1/2} \right) \mathcal{M}(u) \mathcal{Y}_k(2A_1 u),
\]
where $A_1 = \rho_1 \ln \ln (4c^2) + \rho_2 \ln \ln \ln (4c^2)$ and
\[
\mathcal{M}(u) = \begin{cases} 
2A_1 \sqrt{\frac{\pi \ln(4c^2) \ln(\ln(4c^2))}{\rho_1 \ln(\ln(4c^2)) + \rho_2}} u^{1/2}, & \text{for } \rho_2 > -\rho_1 \ln(4c^2), \\
2\Gamma(1/4)A_1 \left( \frac{\ln(\ln(4c^2))^{1/2}}{8\rho_1} \right)^{1/4} u^{3/4}, & \text{for } \rho_2 = -\rho_1 \ln(4c^2),
\end{cases}
\]
and if $\rho_2 < -\rho_1 \ln(4c^2)$, then
\[
P \left\{ \sup_{t \in (0,1)} \frac{\chi^2_b(t)}{w^2_{\rho,\varepsilon}(t)} > u \right\} \sim 2A_2 \left( \prod_{i=k+1}^n (1 - b_i^2)^{-1/2} \right) \rho_1^{-1} Q \sqrt{2\pi \rho_2 u^{1/2}} \psi_k(2A_2 u),
\]
where $A_2 = \rho_2 (\ln(-\rho_2) - \ln(\rho_1) - 1)$ and
\[
Q = -\frac{1}{2t_1 - 1} \ln \left( \frac{e^2}{t_1(t_1 - t_1)} \right), \quad t_1 = 1/2 + \sqrt{1/4 - e^{2 - 4\rho_2 / \rho_1}}.
\]

Next, we consider $B_H(t), t \geq 0$, to be the standard fBm with Hurst index $H \in (0,1)$ and covariance function
\[
\text{Cov}(B_H(s), B_H(t)) = \frac{1}{2} (|s|^{2H} + |t|^{2H} - |s - t|^{2H}), \quad s, t \geq 0.
\]
Denote by $B_H(t) = B_H(t)/t^H, t \in (0,1]$ the normalized standard fBm defined on $(0,1]$. Further, for any $\rho > 0$ and $\varepsilon \in (0,1)$, we define
\[
w^2_{\rho,\varepsilon}(t) = \begin{cases} 
\rho \ln \ln (e^2/t), & \text{for } t \in (0,\varepsilon), \\
\rho \ln \ln (e^2/\varepsilon), & \text{for } t \in [\varepsilon,1].
\end{cases}
\]

We have the following result.

**Corollary 3.5.** Let $\frac{\chi^2_b(t)}{w^2_{\rho,\varepsilon}(t)}, t \in (0,1]$, be a weighted chi-square process with generic process $B_H(t), t \in (0,1]$ and $w^2_{\rho,\varepsilon}$ given in (15). Then, we have, as $u \to \infty$,
\[
P \left\{ \sup_{t \in (0,1)} \frac{\chi^2_b(t)}{w^2_{\rho,\varepsilon}(t)} > u \right\} \sim \left( \prod_{i=k+1}^n (1 - b_i^2)^{-1/2} \right) \left( -\ln(\varepsilon) \right) \left( \ln \ln (e^2/\varepsilon) \rho/2 \right) \frac{\pi}{2}
\times \mathcal{H}_{2H} u^{-\frac{\pi}{2}} \psi_k(\rho \ln \ln (e^2/\varepsilon) u).
\]

4. PROOFS

This section is devoted to the proof of all the results presented in Section 3.

**Proof of Theorem 3.1:** Note that $t^{2-1}(q(t))^{-1}$ is a positive regularly varying function at $\infty$ with index $\kappa = k/2 - 1 + 1/\alpha \geq 0$. Thus, by Potter bound (e.g., [18])
\[
c_1 t^{\kappa - 1} \leq t^{2-1}(q(t))^{-1} \leq c_2 t^{\kappa + 1}, \quad t \geq c_3,
\]
holds for some constants $c_1, c_2, c_3 > 0$, which, together with the fact that $w^2(t) \to \infty$ as $t \to S$, leads to
\[
Q_1 e^{-w^2(t)} \leq (w(t))^{k-2} q(w^2(t)) e^{-w^2(t)} \leq Q_2 e^{-w^2(t)}
\]
for all $t$ approaching $S$, with some positive constants $Q_1, Q_2$. Therefore, if $J_{c,w}(S) < \infty$ holds for some $c > 0$, then, by (16),
\[
I_{\sqrt{3}c_{w}}(S) < \infty.
\]
This together with ii) of Theorem 2.1 yields that
\[
\limsup_{t \to S} \frac{\chi^2_b(t)}{w^2(t)} \leq 3c \text{ a.s.}
\]
Let \( u > Q \)

**Proof of Theorem 3.3:** Without loss of generality, we show the proof only for the case where showing that 

\[
\sup_{t \in E(S)} \frac{\chi^2_b(t)}{w^2(t)} < \infty \quad \text{a.s.}
\]

On the other hand, if \( J_{c,w}(S) = \infty \) for all \( c > 0 \), then, by (16), \( I_{\sqrt{c}w}(S) = \infty \). Thus, by iii) of Theorem 2.1

\[
\limsup_{t \to S} \frac{\chi^2_b(t)}{w^2(t)} \geq c \quad \text{a.s.}
\]

holds for all \( c > 0 \), implying that

\[
\sup_{t \in E(S)} \frac{\chi^2_b(t)}{w^2(t)} = \infty \quad \text{a.s.}
\]

This completes the proof. \( \Box \)

We present next two lemmas which will play key roles in the proof of Theorem 3.3; see [17] for proofs. Denote below \( S \subseteq \mathbb{R} \) to be any fixed interval.

**Lemma 4.1.** Let \( \chi^2_b(t), t \in S \), be a chi-square process with generic centered Gaussian process \( X \) which has a.s. continuous sample paths and variance function denoted by \( \sigma^2_X(t) \). If

\[
\sup_{t \in S} X(t) < \infty \quad \text{a.s.,}
\]

then there exists some positive constant \( Q \) such that for all \( u > Q^2 \) we have

\[
(17) \quad \mathbb{P} \left\{ \sup_{t \in S} \chi^2_b(t) > u \right\} \leq \exp \left( -\frac{(\sqrt{u} - Q)^2}{2 \sup_{t \in S} \sigma^2_X(t)} \right).
\]

**Lemma 4.2.** Let \( \chi^2_b(t), t \in S \), be a chi-square process with the generic centered locally stationary Gaussian process \( X \) which has a.s. continuous sample paths. If further the correlation function of \( X \) satisfies

\[
(18) \quad r(s, t) < 1 \quad \text{for any } s \neq t \in S,
\]

then, for any compact intervals \( S_1, S_2 \subseteq S \) such that \( S_1 \cap S_2 = \emptyset \) we have

\[
\mathbb{P} \left\{ \sup_{t \in S_1} \chi^2_b(t) > u, \sup_{t \in S_2} \chi^2_b(t) > u \right\} \leq \exp \left( -\frac{(2 \sqrt{u} - Q)^2}{2(2 + \eta)} \right).
\]

for all \( u > Q^2 \), with some constant \( Q > 0 \) and \( \eta \in (0, 1) \).

**Proof of Theorem 3.3:** Without loss of generality, we show the proof only for the case where \( T = (0, 1) \).

(i). Let \( \rho > 0 \) be a sufficiently small constant such that

\[
[t_i - \rho, t_i + \rho] \cap [t_j - \rho, t_j + \rho] = \emptyset, \quad \text{for all } i \neq j.
\]

Further, denote \( T_\rho = T \setminus \bigcup_{i=1}^m [t_i - \rho, t_i + \rho] \). It follows from the Bonferroni inequality (e.g., [19]) that

\[
\sum_{i=1}^m p_i(u) + \mathbb{P} \left\{ \sup_{t \in T_\rho} \frac{\chi^2_b(t)}{w^2(t)} > u \right\}
\]

\[
\geq \mathbb{P} \left\{ \sup_{t \in T} \frac{\chi^2_b(t)}{w^2(t)} > u \right\}
\]

\[
\geq \sum_{i=1}^m p_i(u) - \sum_{1 \leq i < j \leq m} \mathbb{P} \left\{ \sup_{t \in [t_i - \rho, t_i + \rho]} \frac{\chi^2_b(t)}{w^2(t)} > u, \sup_{t \in [t_j - \rho, t_j + \rho]} \frac{\chi^2_b(t)}{w^2(t)} > u \right\},
\]

where

\[
p_i(u) = \mathbb{P} \left\{ \sup_{t \in [t_i - \rho, t_i + \rho]} \frac{\chi^2_b(t)}{w^2(t)} > u \right\}.
\]
We first focus on the asymptotics of $p_i(u)$ as $u \to \infty$. Denote
\[ Y(t) = \frac{w(t_1)}{w(t)}X(t), \quad t \in \mathcal{T}. \]

We have
\[ p_i(u) = \mathbb{P} \left\{ \sup_{t \in [t_i - \rho, t_i + \rho]} \sum_{l=1}^n b^2_l Y_l^2(t) > w^2(t_1)u \right\}, \quad 1 \leq i \leq m, \]
where $\{Y_i\}_{i=1}^n$ is a sequence of independent copies of Gaussian process $Y$. It can be shown that, by F1, for any $i = 1, 2, \ldots, m,$
\[ \sigma_Y(t) = \sqrt{\mathbb{E}((Y(t))^2)} = \frac{w(t_1)}{w(t)}, \quad t \in [t_i - \rho, t_i + \rho], \]
attains its maximum which is equal to 1 at the unique point $t_i$, and further
\[ \sigma_Y(t_i + t) = 1 - \frac{a_i}{w(t_1)} |t|^{\beta_i}(1 + o(1)), \quad t \to 0. \]
Moreover, by (3)
\[ 1 - Corr(Y(t_i + t), Y(t_i + s)) = C(t_i)K^2(|t - s|)(1 + o(1)), \quad t \to 0. \]
Consequently, it follows from [4][Theorem 5.2] that, as $u \to \infty$,
\[ p_i(u) \sim \left( \prod_{l=k+1}^n \left( 1 - b_l^2 \right)^{-1/2} \right) M_i(\beta_i, u) \mathcal{Y}_k(w^2(t_1)u), \]
where $\mathcal{Y}_k(\cdot)$ is given in (14) and
\[ M_i(\beta_i, u) = \begin{cases} \frac{2a_i^{-1/\beta_i}(w(t_1))^{1/\beta_i}(C(t_i))^{1/\alpha}}{(1/\beta_i + 1)H_\alpha(q(u))^{-1}u^{-1/\beta_i}}, & \text{for } C1(\beta_i), \\ \mathcal{P}_{\alpha}(w(t_1)C(t_i))^{-1}L^\alpha, & \text{for } C2(\beta_i), \\ 1, & \text{for } C3(\beta_i). \end{cases} \]

In the sequel, we discuss the three scenarios $C1(\beta)$, $C2(\beta)$, $C3(\beta)$ one-by-one.

**C1(\beta)**. Using the fact that $\beta = \max_{i=1}^m \beta_i$, we have that
\[ M_j(\beta_j, u) = o(M_i(\beta_i, u)), \quad u \to \infty \]
for any $i \in K$ and $j \in K^c$. This implies that
\[ \sum_{i=1}^m p_i(u) \sim \sum_{i \in K} p_i(u) \sim \left( \prod_{l=k+1}^n \left( 1 - b_l^2 \right)^{-1/2} \right) M(u) \mathcal{Y}_k(w^2(t_1)u), \]
where $M(\cdot)$ is given in (15). On the other hand, it follows directly from Lemma 4.1 that
\[ \mathbb{P} \left\{ \sup_{t \in \mathcal{T}_u} \frac{\chi^2(t)}{w^2(t)} > u \right\} \leq \exp \left( - \frac{\inf_{t \in \mathcal{T}_u} w^2(t)(\sqrt{u} - Q)^2}{2} \right) \]
holds for all $u > Q^2$, with $Q$ some positive constant. Since further, by F1,
\[ \inf_{t \in \mathcal{T}_u} w^2(t) > w^2(t_1), \]
we have that
\[ \mathbb{P} \left\{ \sup_{t \in \mathcal{T}_u} \frac{\chi^2(t)}{w^2(t)} > u \right\} = o(M(u) \mathcal{Y}_k(w^2(t_1)u)), \quad u \to \infty. \]
Moreover, since for any $i \neq j$

$$\mathbb{P}\left\{ \sup_{t \in [t_i - \rho, t_i + \rho]} \frac{\lambda^2_b(t)}{w^2(t)} > u, \sup_{t \in [t_j - \rho, t_j + \rho]} \frac{\lambda^2_b(t)}{w^2(t)} > u \right\}$$

$$\leq \mathbb{P}\left\{ \sup_{t \in [t_i - \rho, t_i + \rho]} \lambda^2_b(t) > w^2(t_1)u, \sup_{t \in [t_j - \rho, t_j + \rho]} \lambda^2_b(t) > w^2(t_1)u \right\}.$$

we have from Lemma 4.2 that, for all $u$ large,

$$\mathbb{P}\left\{ \sup_{t \in [t_i - \rho, t_i + \rho]} \lambda^2_b(t) > u, \sup_{t \in [t_j - \rho, t_j + \rho]} \lambda^2_b(t) > u \right\} \leq \exp\left( -\frac{2w(t_1)\sqrt{u - Q_{i,j}}^2}{2(2 + 2\eta)} \right), \quad 1 < i < j \leq m,$$

with $Q_{i,j}$'s some positive constants and $\eta \in (0, 1)$. Therefore, as $u \to \infty$,

$$(23) \quad \sum_{1 \leq i < j \leq m} \mathbb{P}\left\{ \sup_{t \in [t_i - \rho, t_i + \rho]} \lambda^2_b(t) > u, \sup_{t \in [t_j - \rho, t_j + \rho]} \lambda^2_b(t) > u \right\} = o\left( M(u) \, \Upsilon_k(w^2(t_1)u) \right).$$

Combining (21)–(23) with (19) we establish the claim of C1($\beta$).

**C2($\beta$).** In this case, we have that (20) holds with

$$M_i(\beta_i, u) = \begin{cases} \mathcal{P}_\alpha^{a_i(w(t_1)C(t_i))^{-1}} \mathcal{L}^a, & i \in K \\ 1, & i \in K^c. \end{cases}$$

Consequently,

$$\sum_{i=1}^m p_i(u) \sim \left( \prod_{i=k+1}^n (1 - b_i^2)^{-1/2} \right) \left( \sum_{i \in K} \mathcal{P}_\alpha^{a_i(w(t_1)C(t_i))^{-1}} \mathcal{L}^a + \#K^c \right) \Upsilon_k(w^2(t_1)u).$$

Note that (22) and (23) still hold. Similarly as the case C1($\beta$), we establish the claim of C2($\beta$).

**C3($\beta$).** In this case, we have that (20) holds with

$$M_i(\beta_i, u) = 1, \quad 1 \leq i \leq m.$$

Consequently,

$$\sum_{i=1}^m p_i(u) \sim m \left( \prod_{i=k+1}^n (1 - b_i^2)^{-1/2} \right) \Upsilon_k(w^2(t_1)u).$$

Similarly as before, the claim of C3($\beta$) follows.

(ii) By F2 we have for any sufficiently small $\varepsilon > 0$ it holds that

$$\inf_{t \in \tau_{c_1}} w(t) > w(c_1), \quad \text{with } \tau_{c_1} = \tau \setminus \bigcup_{i=1}^m [c_i - \varepsilon, d_i + \varepsilon].$$

Similarly to (19) we have

$$(24) \quad \sum_{i=1}^m \mathbb{P}\left\{ \sup_{t \in [c_i, \varepsilon_i, d_i, \varepsilon_i]} \frac{\lambda^2_b(t)}{w^2(t)} > u \right\} + \mathbb{P}\left\{ \sup_{t \in \tau_{c_1}} \frac{\lambda^2_b(t)}{w^2(t)} > u \right\} \geq \mathbb{P}\left\{ \sup_{t \in \tau} \frac{\lambda^2_b(t)}{w^2(t)} > u \right\} \geq \sum_{i=1}^m \mathbb{P}\left\{ \sup_{t \in [c_i, d_i]} \lambda^2_b(t) > u \right\} - \sum_{1 \leq i < j \leq m} \mathbb{P}\left\{ \sup_{t \in [c_i, d_i]} \lambda^2_b(t) > u, \sup_{t \in [c_j, d_j]} \lambda^2_b(t) > u \right\}.$$
Next, we have from $F_2$ that for $1 \leq i \leq m$

$$
P \left\{ \sup_{t \in [c_i, d_i]} \frac{\chi^2_b(t)}{w^2(t)} > u \right\} = P \left\{ \sup_{t \in [c_i, d_i]} \chi^2_b(t) > w^2(c_1)u \right\},$$

$$
P \left\{ \sup_{t \in [c_i - \epsilon, d_i + \epsilon]} \frac{\chi^2_b(t)}{w^2(t)} > u \right\} \leq P \left\{ \sup_{t \in [c_i - \epsilon, d_i + \epsilon]} \chi^2_b(t) > w^2(c_1)u \right\}.
$$

It is noted that the result in Theorem 2.1 of [10] also holds when $g(t) = 0$. Thus, it follows from that result, as $u \to \infty$,

$$
P \left\{ \sup_{t \in [c_i, d_i]} \frac{\chi^2_b(t)}{w^2(t)} > w^2(c_1)u \right\} \sim \prod_{j=k+1}^n (1 - b_j)^{-1/2} \mathcal{H}_a \int_{c_i}^{d_i} (C(t))^{1/\alpha} dt (q(w^2(c_1)u))^{-1} \chi_{2k}(w^2(c_1)u),
$$

$$
P \left\{ \sup_{t \in [c_i - \epsilon, d_i + \epsilon]} \frac{\chi^2_b(t)}{w^2(t)} > w^2(c_1)u \right\} \sim \prod_{j=k+1}^n (1 - b_j)^{-1/2} \mathcal{H}_a \int_{c_i - \epsilon}^{d_i + \epsilon} (C(t))^{1/\alpha} dt (q(w^2(c_1)u))^{-1} \chi_{2k}(w^2(c_1)u).
$$

Moreover, since

$$
P \left\{ \sup_{t \in [c_i, d_i]} \frac{\chi^2_b(t)}{w^2(t)} > u, \ sup_{t \in [c_j, d_j]} \frac{\chi^2_b(t)}{w^2(t)} > u \right\} = P \left\{ \sup_{t \in [c_i, d_i]} \chi^2_b(t) > w^2(c_1)u, \ sup_{t \in [c_j, d_j]} \chi^2_b(t) > w^2(c_1)u \right\},
$$

we have from Lemma 4.2 that, for all $u$ large,

$$
P \left\{ \sup_{t \in [c_i, d_i]} \frac{\chi^2_b(t)}{w^2(t)} > u, \ sup_{t \in [c_j, d_j]} \frac{\chi^2_b(t)}{w^2(t)} > u \right\} \leq \exp \left( \frac{(2w(c_1)\sqrt{u} - Q_{i,j})^2}{2(2 + 2\eta)} \right), \ 1 \leq i < j \leq m,
$$

with $Q_{i,j}$’s some positive constants and $\eta \in (0, 1)$. This implies that

$$
\sum_{1 \leq i < j \leq m} P \left\{ \sup_{t \in [c_i, d_i]} \frac{\chi^2_b(t)}{w^2(t)} > u, \ sup_{t \in [c_j, d_j]} \frac{\chi^2_b(t)}{w^2(t)} > u \right\} = o \left( (q(w^2(c_1)u))^{-1} \chi_{2k}(w^2(c_1)u) \right), \ u \to \infty.
$$

Moreover, Lemma 4.1 gives that

$$
P \left\{ \sup_{t \in T} \frac{\chi^2_b(t)}{w^2(t)} > u \right\} = o \left( (q(w^2(c_1)u))^{-1} \chi_{2k}(w^2(c_1)u) \right), \ u \to \infty.
$$

Consequently, by letting $\epsilon \to 0$ we conclude that the claim in (ii) is established. This completes the proof. \hfill \Box

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**References**


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